NESTED QUANTIFIERS

Section 1.5

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REVIEW EXERCISES

Translating the following sentences in logic:

- Every freshman at the College is taking some Computer Science course.
- All lions are fierce.
- Some lions do not drink coffee.
- Some fierce creatures do not drink coffee.

SECTION SUMMARY

- Nested Quantifiers
- Order of Quantifiers
- Translating from Nested Quantifiers into English
- Translating Mathematical Statements into Statements involving Nested Quantifiers.
- Translated English Sentences into Logical Expressions.
- Negating Nested Quantifiers.

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NESTED QUANTIFIERS

- Nested quantifiers are quantifiers where one quantifier is within the scope of another.

Example: "Every real number has an inverse" is

$\forall x\, \exists y(x + y = 0)$

where the domains of x and y are the real numbers.

- We can also think of nested propositional functions:

$\forall x\, \exists y(x + y = 0)$ can be viewed as

$\forall x\, Q(x)$ where $Q(x)$ is $\exists y\, P(x, y)$ where $P(x, y)$ *is* $(x + y = 0)$

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NESTED QUANTIFIERS

Write the following statements in English, using the predicate $S(x, y)$: "x shops in y", where x represents people and y represents stores:

- (a) $\forall y\, S(Margaret, y)$.
- (b) $\exists x \forall y\, S(x, y)$.

Solution:

(a) Margaret shops in every store.

(b) There is a person who shops in every store.

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MORE EXAMPLES

Consider these lines of code from a Python program:

```
if (not(x !=0 and y/x < 1) or x==0):
    print (True)
else:
    print (False)
```

Express the code in this statement as a compound statement using the logical connectives ¬, ∨, ∧, →, and these predicates

$E(x)$: $x = 0$

$L(x, y)$: $y/x < 1$

$A(z)$: "print z"

where x and y are integers and z is a Boolean variable (with values True and False).

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MORE EXAMPLES

Suppose that the universe for x and y is $\{1, 2, 3\}$. Also, assume that $P(x, y)$ is a predicate that is true in the following cases, and false otherwise: $P(1, 3)$, $P(2, 1)$, $P(2, 2)$, $P(3, 1)$, $P(3, 2)$, $P(3, 3)$. Determine whether each of the following is true or false:

(a) $\forall y \exists x\, (x \neq y \wedge P(x, y))$.

(b) $\forall x \exists y\, (x \neq y \wedge \neg P(x, y))$.

(c) $\forall y \exists x\, (x \neq y \wedge \neg P(x, y)$.

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THINKING OF NESTED QUANTIFICATION

- Nested Loops
  - To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :
    - At each step, loop through the values for y.
    - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.
    - $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.
  - To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
    - At each step, loop through the values for y.
    - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
    - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.
    - $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.
- If the domains of the variables are infinite, then this process can not actually be carried out.

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ORDER OF QUANTIFIERS

The order of the quantifiers is important, unless all the quantifiers are the same.

Examples:

1. Let $P(x,y)$ be the statement "$x + y = y + x$." Assume that U is the real numbers. Then $\forall x \forall y P(x,y)$ and $\forall y \forall x P(x,y)$ have the same truth value.

2. Let $Q(x,y)$ be the statement "$x + y = 0$." Assume that U is the real numbers. Then

$$\forall x \exists y\, Q(x,y)$$

is true, *and*

$$\exists y \forall x\, Q(x,y)$$

is false.

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QUANTIFICATIONS OF TWO VARIABLES

Statement	When True?	When False
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	*P*(*x*, *y*) is true for every pair *x*, *y*.	There is a pair *x*, *y* for which *P*(*x*, *y*) is false.
$\forall x \exists y P(x, y)$	For every *x* there is a *y* for which *P*(*x*, *y*) is true.	There is an x such that *P*(*x*,*y*) is false for every *y*.
$\exists x \forall y P(x, y)$	There is an *x* for which *P*(*x*, *y*) is true for every *y*.	For every *x* there is a *y* for which *P*(*x*, *y*) is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair *x*, *y* for which *P*(*x*, *y*) is true.	*P*(*x*, *y*) is false for every pair *x*, *y*

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QUANTIFICATIONS OF MORE VARIABLES

Let $Q(x, y, z)$ be the statement "$x + y = z$." What are the truth values of the statements $\forall x \forall y \exists z\, Q(x, y, z)$ and $\exists z \forall x \forall y\, Q(x, y, z)$, where the domain of all variables consists of all real numbers.

Solution: Suppose that x and y are assigned values. Then, there exists a real number z such that $x + y = z$. Consequently, the quantification

$$\forall x \forall y \exists z\, Q(x, y, z),$$

which is the statement "For all real numbers x and for all real numbers y there is a real number z such that $x + y = z$," is true.

$$\exists z \forall x \forall y\, Q(x, y, z)?$$

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TRANSLATING MATHEMATICAL STATEMENTS INTO PREDICATE LOGIC

Example : Translate "The sum of two positive integers is always positive" into a logical expression.

Solution:

1. Rewrite the statement to make the implied quantifiers and domains explicit:

   "For every two integers, if these integers are both positive, then the sum of these integers is positive."

2. Introduce the variables x and y, and specify the domain, to obtain:

   "For all positive integers x and y, $x + y$ is positive."

3. The result is:

   $$\forall x \forall y ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$$

   where the domain of both variables consists of all integers

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TRANSLATING NESTED QUANTIFIERS INTO ENGLISH

Example 1: Translate the statement

$$\forall x \ (C(x) \lor \exists y \ (C(y) \land F(x, y)))$$

where $C(x)$ is "x has a computer," and $F(x,y)$ is "x and y are friends," and the domain for both x and y consists of all students in your school.

Solution: Every student in your school has a computer or has a friend who has a computer.

Example 2: Translate the statement

$$\exists x \forall y \forall z \ ((F(x, y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z))$$

Solution: There exists some student none of whose friends are also friends with each other. （有个同学，他的朋友之间不是朋友）

TRANSLATING ENGLISH INTO LOGICAL EXPRESSIONS EXAMPLE

Example: Express the statement "If a person is female and is a parent, then this person is someone's mother" as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.

Solution:

- For every person x, if person x is female and person x is a parent, then there exists a person y such that person x is the mother of person y.
- $F(x)$ to represent "x is female," $P(x)$ to represent "x is a parent," and $M(x, y)$ to represent "x is the mother of y."
- $\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$
- $\forall x \exists y((F(x) \wedge P(x)) \rightarrow M(x, y))$.

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TRANSLATING ENGLISH INTO LOGICAL EXPRESSIONS EXAMPLE

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f" and $Q(f,a)$ be "f is a flight on a."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \, \forall a \, \exists f \; (P(w,f) \land Q(f,a))$$

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NEGATING NESTED QUANTIFIERS

Example 1: Recall the logical expression:

$$\exists w\ \forall a\ \exists f\ \ (P(w,f\,)\ \wedge\ Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution:

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

$$\neg\exists w\ \forall a\ \exists f\ \ (P(w,f\,)\ \wedge\ Q(f,a))$$

Part 3: Can you translate the result back into English?

Solution:

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