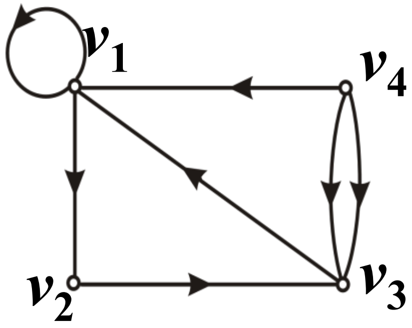
↳ Counting Directed Paths via Adjacency Matrices

- **Theorem 6.4:** Let *A* be the adjacency matrix of the n-order directed graph ***D***. Then, the elements of $A^l (l \geq 1)$:
  - $a_{ij}^{(l)}$ are the number of *paths of length l* from vertex v_i to vertex v_j in ***D***.
  - $a_{ii}^{(l)}$ is the number of *cycles of length l* starting and ending at vertex v_i .
  - $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^{(l)}$ is the *total number of paths of length l* (including cycles) in ***D***.
  - $\sum_{i=1}^{n} a_{ii}^{(l)}$ is the *total number of cycles of length l* in ***D***.

- **Corollary:** Let $B_l=A+A^2+\ldots+A^l (l\geq 1)$, Then, the elements :
  - $b_{ij}^{(l)}$ are the **number of paths** (including cycles) of length *less than or equal to l* from vertex v_i to vertex v_j in ***D***.
  - $b_{ii}^{(l)}$ is the **number of cycles** in *D* whose length from v_i to v_i *less than or equal to l*.
  - $\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^{(l)}$ is the **number of paths** (including cycles) in ***D*** whose length is *less than or equal to l*.
  - $\sum_{i=1}^{n} b_{ii}^{(l)}$ is the **number of cycles** in ***D*** whose length is *less than or equal to l*.

↳ Counting Directed Paths via Adjacency Matrices (e.g.)



$$A=\begin{bmatrix}1&1&0&0\\0&0&1&0\\1&0&0&0\\1&0&2&0\end{bmatrix}$$

$$A^2=\begin{bmatrix}1&1&1&0\\1&0&0&0\\1&1&0&0\\3&1&0&0\end{bmatrix}$$

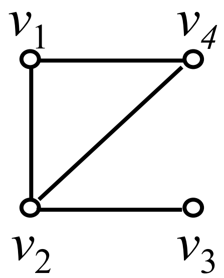
$$A^3=\begin{bmatrix}2&1&1&0\\1&1&0&0\\1&1&0&0\\3&3&1&0\end{bmatrix}$$

$$A^4=\begin{bmatrix}3&2&1&0\\1&1&0&0\\2&1&1&0\\4&3&1&0\end{bmatrix}$$

- There is 1 path of length 3 from v_1 to v_2.
- There is 1 path of length 3 from v_1 to v_3.
- There are 2 cycles of length 3 from v_1 to itself.
- There are a total of 15 paths of length 3 in D, of which 3 are cycles.

↳ Adjacency Matrix of a Undirected Graph

- Let $G=\langle V,E\rangle$ be an undirected simple graph, where $V=\{v_1,v_2,\ldots,v_n\}$. Let $a_{ij}^{(1)}$ denote the number of edges between vertices v_i and v_j. The matrix $(a_{ij})_{n\times n}$ is called the ***adjacency matrix of G***, denoted as $A(G)$.
- **Example**: Write the adjacency matrix of an undirected graph, and find the number of paths of length **3** from v_i to v_2 and the number of cycles of length 3 from v_1 to v_1.



$$A=\begin{bmatrix}0&1&0&1\\1&0&1&1\\0&1&0&0\\1&1&0&0\end{bmatrix}\qquad A^2=\begin{bmatrix}2&1&1&1\\1&3&0&1\\1&0&1&1\\1&1&1&2\end{bmatrix}\qquad A^3=\begin{bmatrix}2&3&1&3\\3&2&3&4\\1&3&0&1\\3&4&1&2\end{bmatrix}$$

There are **3 paths** of length 3 from v_1 to v_2: $v_1v_2v_1v_2$, $v_1v_2v_3v_2$, $v_1v_4v_1v_2$.

There are **2 cycles** of length 3 from v_1 to v_1: $v_1v_2v_4v_1$, $v_1v_4v_2v_1$.

- 6.3.1 Incidence Matrix of an Undirected Graph
- 6.3.2 Incidence Matrix of a Directed Acyclic Graph
- 6.3.3 Adjacency Matrix of a Directed and Undirected Graph
  The Number of Paths and Cycles in a Graph
- **6.3.4 Reachability Matrix of a Graph**

- Let the graph (either undirected or directed) $G=<V,E>$, $V=\{v_1, v_2, ..., v_n\}$,

  let $$p_{ij} = \begin{cases} 1, & v_i \text{ can reach } v_j \\ 0, & \text{otherwise} \end{cases}$$

  The matrix $(p_{ij})_{n\times n}$ is called the ***reachability matrix*** of G, denoted as $P(G)$, or simply P.

- **Property:**

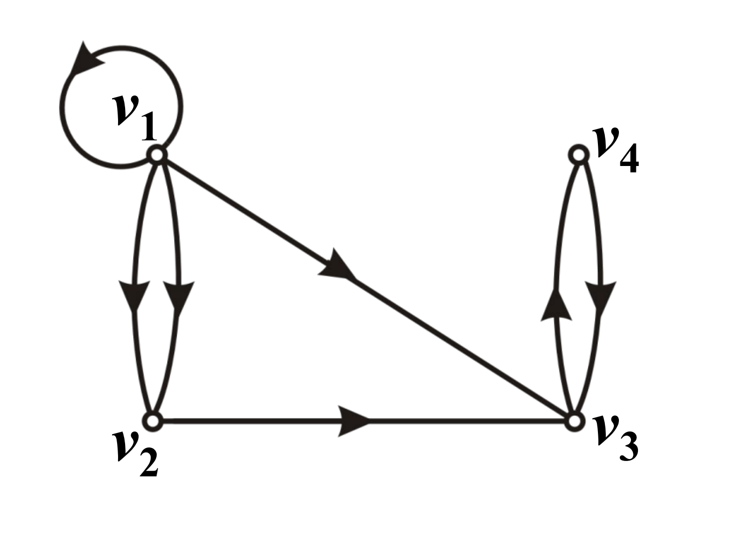
  (1) All the elements on the *main diagonal* of $P(G)$ are 1.

  (2) The reachability matrix of an undirected graph is *symmetric*.

  (3) An *undirected graph* G is **connected** if and only if all elements of $P(G)$ are 1. A *directed graph* D is **strongly connected** if and only if all elements of $P(D)$ are 1.

  (4) For an *n-order graph*, $p_{ij}=1 \Leftrightarrow b_{ij}^{(n-1)} > 0$, $i\neq j$.

- **Example:**



(1) How many paths of length 3 are there from v_1 to v_4, and from v_4 to v_1 ?

(2) How many cycles of length 1, 2, 3, and 4 are there from v_1 to itself?

(3) How many paths of length 4 are there in total ? How many of them are cycles ?

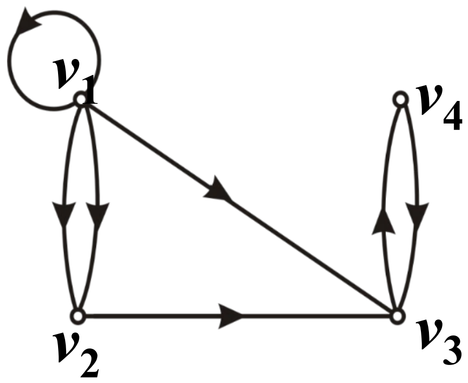
(4) How many cycles of length less than or equal to 4 are there in total ?

(5) Write the reachability matrix of *D*, and is *D* strongly connected?

- **Solution ?**

↳ Reachability matrix of Graph(e.g.)

- **Solution:**



$$A=\begin{bmatrix}1&2&1&0\\0&0&1&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\quad A^2=\begin{bmatrix}1&2&3&1\\0&0&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$A^3=\begin{bmatrix}1&2&4&3\\0&0&1&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\quad A^4=\begin{bmatrix}1&2&6&4\\0&0&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix}\quad P=\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&1&1\end{bmatrix}$$

(1) There are **3** paths of length **3** from v_1 to v_4. There are 0 paths of length 3 from v_4 to v_1.

(2) There are 1 cycles of length 1, 2, 3, and 4 from v_1 to itself.

(3) There are 16 paths of length 4, of which 3 are cycles.

(4) There are 8 cycles of length less than or equal to 4.

(5) Reachability Matrix: $P(G)$

Objective :

Key Concepts :

Discrete Mathematics 2025 Spring

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6.4 Several Special Types of Graphs

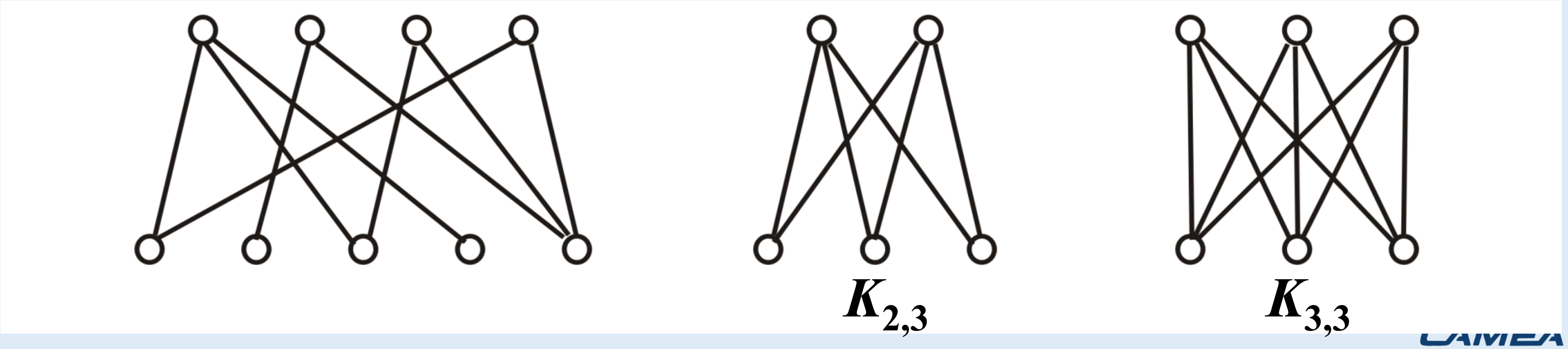
- **6.4.1 Bipartite Graphs**

  Necessary and sufficient conditions for a graph to be bipartite matching, maximal matching, maximum matching, complete matching, perfect matching
- **6.4.2 Eulerian Graphs**

  Eulerian circuits (paths) and their necessary and sufficient conditions for existence
- **6.4.3 Hamiltonian Graphs**

  Hamiltonian circuits (paths) and the necessary and sufficient conditions for their existence
- **6.4.4 Planar Graphs**

- Let G=⟨V,E⟩,G=⟨V,E⟩ be an undirected graph. If it is possible to partition ***V*** into two sets V_1 and V_2 such that: $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \varnothing$, Each edge in ***G*** has one endpoint in V_1 and the other in V_2, then ***G*** is called a ***bipartite graph***, denoted as $<V_1, V_2, E>$, and V_1 and V_2 are called ***complementary vertex subsets***.
  - Furthermore, if ***G*** is a simple graph and every vertex in V_1 is adjacent to every vertex in V_2, then G is called a ***complete bipartite graph***, denoted as $K_{r,s}$, where $r = |V_1|$ and $s = |V_2|$.



- **Theorem 6.5:** An undirected graph $G=<V,E>$ is a bipartite graph if and only if it contains no odd-length cycles.
- **Proof:**

(1) **Necessity:** Let $G=<V_1,V_2,E>$ be a bipartite graph. Each edge can only connect V_1 to V_2 or V_2 to V_1, so any cycle in G must have even length.

(2) **Sufficiency:** Assume G has at least one edge and is connected. Take any vertex u, and define:

$V_1=\{v \mid v \in V$ and the distance from v to u is even$\}$

$V_2=\{v \mid v \in V$ and the distance from v to u is odd$\}$

Then, $V_1 \cup V_2=V$, $V_1 \cap V_2=\varnothing$.

■ Proof:

(2) **Sufficiency:** Assume G has at least one edge and is connected. Take any vertex u, and define:

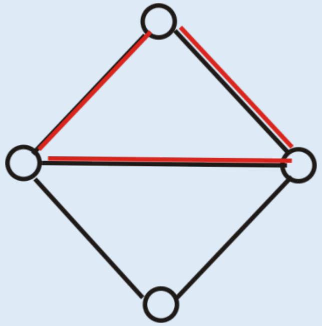
$V_1=\{v \mid v \in V$ and the distance from v to u is even$\}$

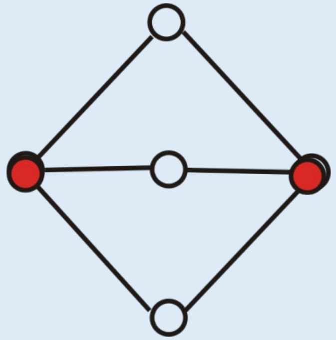
$V_2=\{v \mid v \in V$ and the distance from v to u is odd$\}$

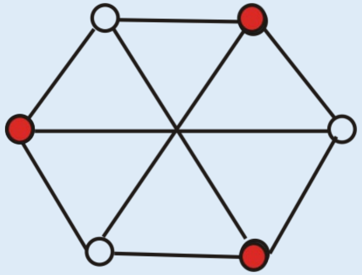
Then, $V_1 \cup V_2=V$, $V_1 \cap V_2=\varnothing$.

- First, we prove that no two vertices in V_1 are adjacent. Suppose there exist $s,t \in V_1$ such that $e=(s,t) \in E$. Let Γ_1 and Γ_2 be the shortest paths from u to s and u to t, respectively. Then, $\Gamma_1 \cup e \cup \Gamma_2$ forms a cycle of odd length, which contradicts the assumption.
- Similarly, we can prove that no two vertices in V_2 are adjacent.

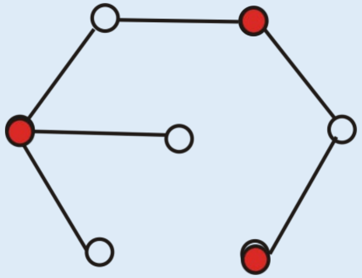
↳ Cycle Characterization of Bipartite Graphs(e.g.)

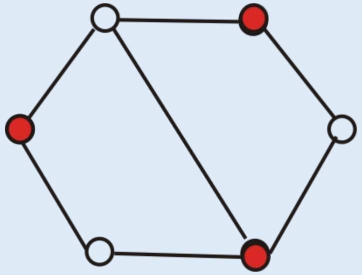


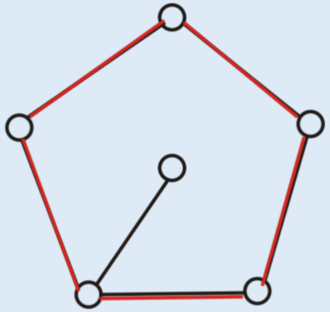




Non-bipartite graph



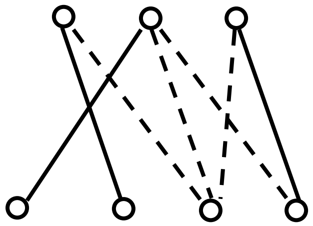




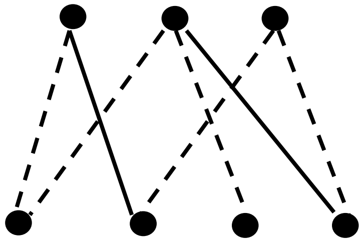
Non-bipartite graph

- **Definition 6.11:** Let $G=<V_1,V_2,E>$ be a bipartite graph, where $E' \subseteq E$. If the edges in E' are pairwise non-adjacent, then E' is called a ***matching*** in G. If adding any edge to E' results in a set of edges that is no longer a matching, then E' is called a ***maximal matching*** in G. The matching in G with the maximum number of edges is called the ***maximum matching*** of G.
    - Moreover, suppose $|V_1| \leq |V_2|$ and E' is a matching in G. If $|E'|=|V_1|$, then E' is called a ***complete matching*** from V_1 to V_2 .
    - When $|V_1|=|V_2|$, a complete matching is called a ***perfect matching***.

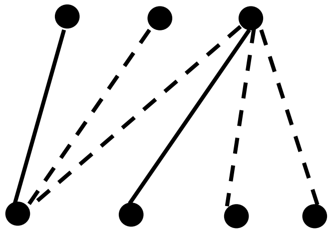
↳ Matchings and Complete Matchings in Bipartite Graphs(e.g.)



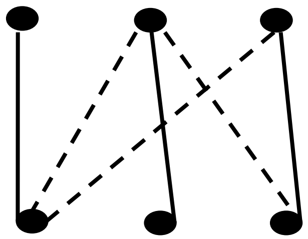
Maximum Matching and Complete Matching



Maximal Matching



Maximum Matching



Perfect Matching

- **Theorem 6.6 (Hall's Theorem):**

  Let $G=<V_1,V_2,E>$ be a bipartite graph with $|V_1|=|V_2|$.
  There ***exists a complete matching*** from V_1 to V_2 in G if and only if, for any k (where $1 \le k \le |V_1|$), the set of k vertices in V_1 is adjacent to at least k vertices in V_2 (the ***distinctness condition***).

- **Theorem 6.7:**

  Let $G=<V_1,V_2,E>$ be a bipartite graph with $|V_1| \le |V_2|$. If there exists a positive integer t such that each vertex in V_1 is connected to at least t edges, and each vertex in V_2 is connected to at most t edges (the **t-condition**), then there ***exists a complete matching*** from V_1 to V_2 in G.

- **Example:**

  A middle school has three extracurricular activity groups: the Math Group, the Computer Group, and the Biology Group. There are five students: Zhao, Qian, Sun, Li, and Zhou. In each of the following three cases, determine whether it is possible to select three students to serve as group leaders, one for each group:

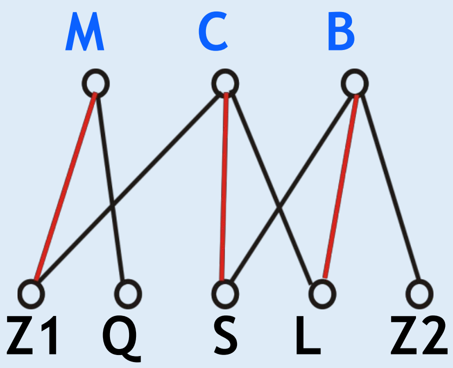
(1) Zhao and Qian are members of the Math Group; Zhao, Sun, and Li are members of the Computer Group; Sun, Li, and Zhou are members of the Biology Group.

(2) Zhao is a member of the Math Group; Qian, Sun, and Li are members of the Computer Group; Qian, Sun, Li, and Zhou are members of the Biology Group.

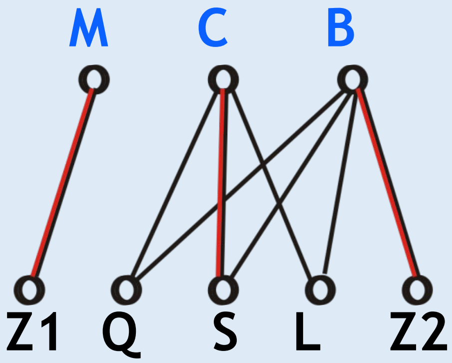
(3) Zhao is a member of both the Math and Computer Groups; Qian, Sun, Li, and Zhou are members of the Biology Group.

↳ Application Examples of Complete Matchings

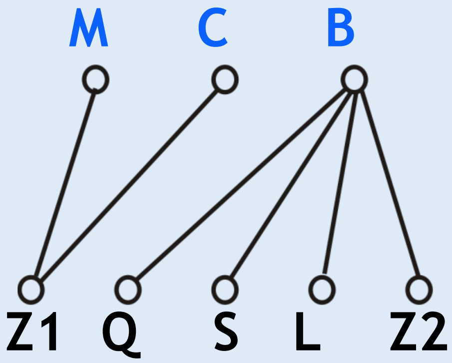
M: Math Group
C: Computer Group
B: Biology Group
Z1: Zhao
Q: Qian
S: Sun
L: Li
Z2: Zhou



(1)



(2)



(3)

A complete matching corresponds to a feasible assignment.

(1) and (2) admit complete matchings, with multiple possible assignments.

(3) does not satisfy the distinctness condition, so no complete matching exists.