Discrete Mathematics 2025 Spring

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■ 6.2.1 Paths and Circuits

Elementary Paths (Circuits) and Simple Paths (Circuits)

■ 6.2.2 Connectivity and Connectedness in Undirected Graphs

Connected Graphs and Connected Components

Shortest Paths and Distances

Vertex Cut Sets, Cut Vertices, Edge Cut Sets, and Bridges

Vertex Connectivity and Edge Connectivity

■ 6.2.3 Connectivity and Classification in Directed Graphs

Reachability

Weak Connectivity, Unilateral Connectivity, and Strong Connectivity

Shortest Paths and Distances

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- **Definition 6.8:** Given a graph G=<V,E>(either undirected or directed), an alternating sequence of vertices and edges in G $\Gamma = v_0 e_1 v_1 e_2 \ldots e_l v_l$.

(1) $\forall i(1 \le i \le l)$, $e_i = (v_{i-1}, v_i)$. Γ is called a ***path*** from v_0 to v_l, v_0 and v_l are called the **starting point** and **ending point** of the path, respectively, and l is the **length** of the path. **If** $v_0 = v_l$, Γ the path is called a ***circuit*** (or **cycle**).

(2) If all the vertices in a path or circuit are distinct (except for $v_0 = v_l$ in the case of a circuit), it is called an ***elementary path*** or **simple path** (and an **elementary circuit** or **simple cycle**). A cycle of odd length is called an **odd cycle**, and a cycle of even length is called an **even cycle**.

(3) If all edges in a path or circuit are distinct, it is called a ***simple path*** (or **simple circuit**); otherwise, it is called a **non-simple** or **complex path** (or **complex circuit**).

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Representations of a Path or Circuit

① According to the definition, using an ***alternating sequence*** of vertices and edges: $\Gamma = v_0 e_1 v_1 e_2 \dots e_l v_l$.

② Using a ***sequence of edges***: $\Gamma = e_1 e_2 \dots e_l$.

③ In a simple graph, using a ***sequence of vertices***: $\Gamma = v_0 v_1 \dots v_l$

Properties of Circuits

- In an **undirected graph**, a *circuit of length 1* is formed by a *loop* (an edge connecting a vertex to itself). A *circuit of length 2* is formed by *two parallel edges* between the same pair of vertices. In an **undirected simple graph**, all *circuits have length ≥ 3*.
- In a **directed graph**, a circuit of length **1** is also formed by a **loop**. In a **directed simple graph**, all circuits have length ≥ **2**.

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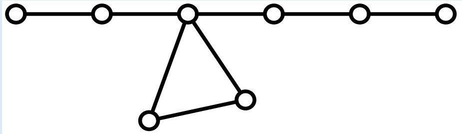
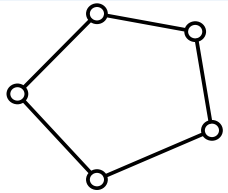
↳ Elementary paths (or circuit) ⊂ simple paths (or circuit)

- Every elementary path (or circuit) is a simple path (or circuit), but the converse is not necessarily true.
- **Example:**

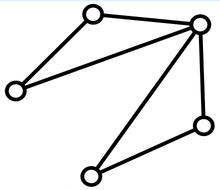
Elementary Path



Non-elementary simple path



Elementary Cycle



Non-elementary simple cycle

- **6.2.1 Paths and Circuits**

  Elementary Paths (Circuits) and Simple Paths (Circuits)

- **6.2.2 Connectivity and Connectedness in Undirected Graphs**

  Connected Graphs and Connected Components

  Shortest Paths and Distances

  Vertex Cut Sets, Cut Vertices, Edge Cut Sets, and Bridges

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- **6.2.3 Connectivity and Classification in Directed Graphs**

  Reachability

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- **Let G=<V,E>** be an undirected graph, and let $u,v \in V$
  - ***Connectivity between u and v***: Vertex u is said to be **connected** to vertex v if there exists a path between them. By convention, every vertex is considered to be connected to itself.
  - ***Connected graph:*** A graph in which every pair of vertices is connected. A **trivial graph** (with only one vertex) is considered connected.
  - ***Connectivity relation***: $R=\{\langle u,v\rangle \mid u,v \in V$ **and u is connected to v**$\}$. R is an **equivalence relation**.
  - ***Connected component***: The subgraph induced by each equivalence class of V under the relation R is called a **connected component** of G. Suppose $V/R=\{V_1,V_2,\ldots,V_k\}$, then the connected components of G are the subgraphs $G[V_1],G[V_2],\ldots,G[V_k]$.
  - ***Number of connected components***: $p(G)=k$

    G is a connected graph $\Leftrightarrow p(G)=1$.

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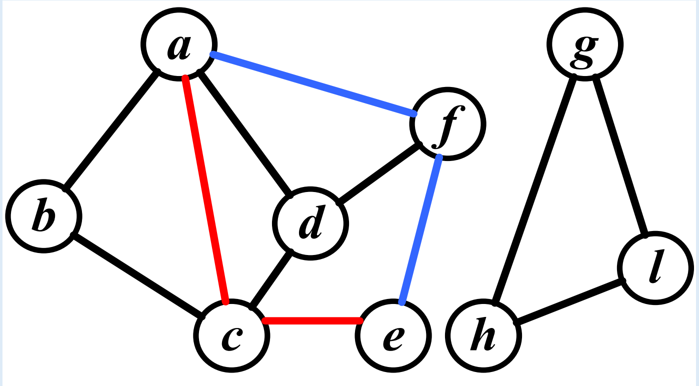
- ***Shortest path between u and v***: A path of the **shortest length** between vertices ***u and v***, assuming ***u and v*** are connected.
- ***Distance between u and v d(u,v)***: The **length** of the shortest path between ***u and v***. If ***u and v*** are **not connected**, define $d(u,v)=\infty$.
- **Property:**
  (1) $d(u,v)\geq 0$, and $d(u,v)=0 \Leftrightarrow u=v$
  (2) $d(u,v)=d(v,u)$
  (3) $d(u,v)+d(v,w)\geq d(u,w)$



- **For example:** The shortest path between ***a*** and ***e*** is shown in the figure on the right: ***ace***, ***afe***. $d(a,e)=2$, $d(a,h)=\infty$

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- Let undirected graph $G=<V,E>$, $v\in V$, $e\in E$, $V'\subseteq V$, $E'\subseteq E$.
  - $G-v$: The graph obtained by **removing vertex v and all edges** incident to it from G.
  - $G-V'$: The graph obtained by **removing all vertices in V'** and their incident edges from G.
  - $G-e$: The graph obtained by **removing edge e** from G.
  - $G-E'$: The graph obtained by **removing all edges in E'** from G.
- **Definition 6.9:** Let undirected graph $G=<V,E>$, $V'\subset V$, If $p(G-V')>p(G)$, then V' is called a ***vertex cut set*** of G. if $\{v\}$ is vertex cut set, then v is called a ***cut vertex***.
  - Let $E'\subseteq E$, if $p(G-E')>p(G)$, then E' is called an ***edge cut set*** **of** G. If $\{e\}$ is edge cut set , then e s called a ***cut edge*** or a ***bridge***.

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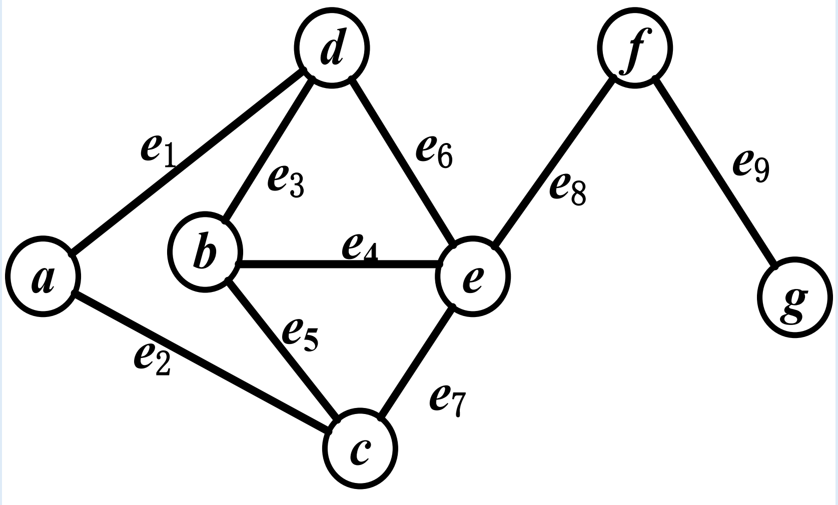
↳ Vertex Cuts and Edge Cuts in Undirected Graphs(e.g.)



Cut vertex: e, f

Vertex cut set : $\{e\}, \{f\}, \{c, d\}$

Bridge: e_8, e_9

Edge cut set :$\{e_8\}, \{e_9\}, \{e_1, e_2\}, \{e_1, e_3, e_6\}, \{e_1, e_3, e_4, e_7\}$

- **Notes:**

(1) The *complete graph* K_n has no vertex cut set.

(2) An *n-vertex null graph* has neither a vertex cut set nor an edge cut set.

(3) If G is connected and E' is an edge cut set, then $p(G-E')=2$.

(4) If G is connected and V' is a vertex cut set, then $p(G-V')\geq 2$.

6.2.2 Connectivity and Connectedness in Undirected

↳ The correct minimum cut problem

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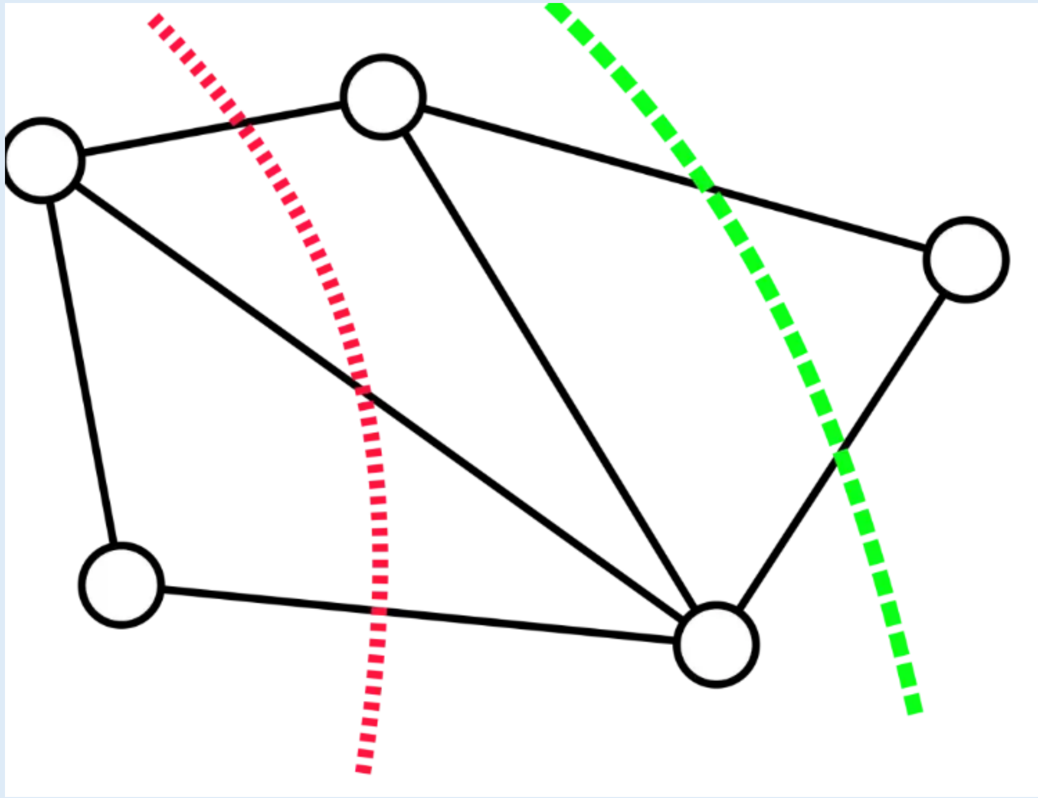

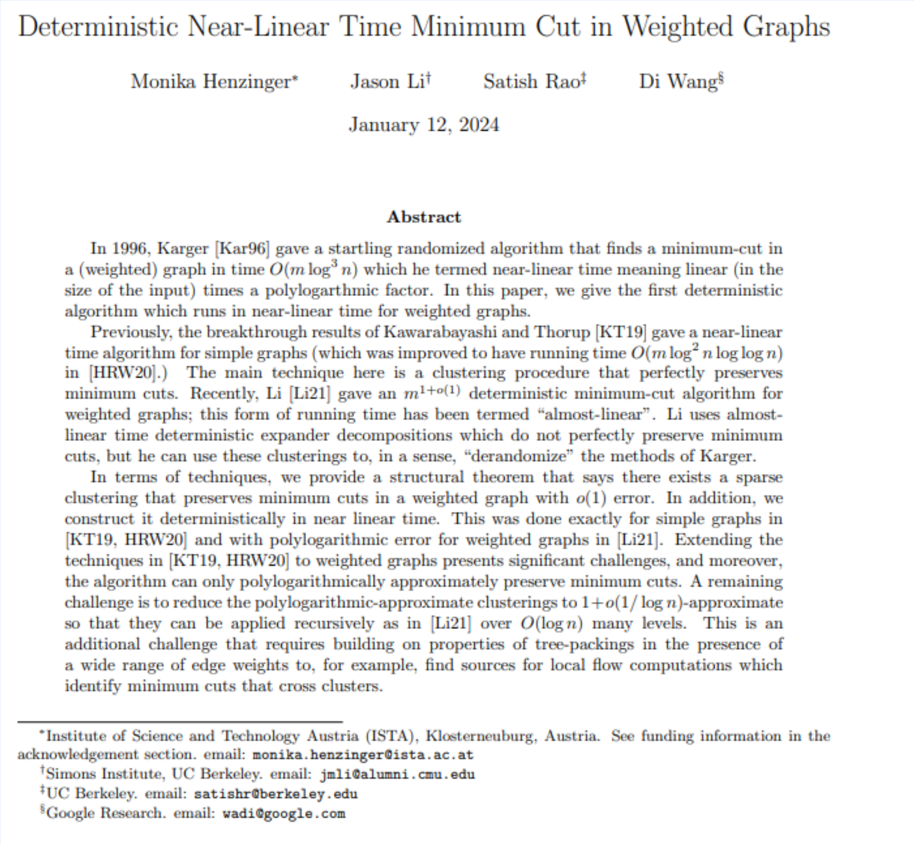

Deterministic Near-Linear Time Minimum Cut in Weighted Graphs

Monika Henzinger* Jason Li† Satish Rao‡ Di Wang§

January 12, 2024

Abstract

In 1996, Karger [Kar96] gave a startling randomized algorithm that finds a minimum-cut in a (weighted) graph in time $O(m \log^3 n)$ which he termed near-linear time meaning linear (in the size of the input) times a polylogarthmic factor. In this paper, we give the first deterministic algorithm which runs in near-linear time for weighted graphs.

Previously, the breakthrough results of Kawarabayashi and Thorup [KT19] gave a near-linear time algorithm for simple graphs (which was improved to have running time $O(m \log^2 n \log\log n)$ in [HRW20].) The main technique here is a clustering procedure that perfectly preserves minimum cuts. Recently, Li [Li21] gave an $m^{1+o(1)}$ deterministic minimum-cut algorithm for weighted graphs; this form of running time has been termed "almost-linear". Li uses almost-linear time deterministic expander decompositions which do not perfectly preserve minimum cuts, but he can use these clusterings to, in a sense, "derandomize" the methods of Karger.

In terms of techniques, we provide a structural theorem that says there exists a sparse clustering that preserves minimum cuts in a weighted graph with $o(1)$ error. In addition, we construct it deterministically in near linear time. This was done exactly for simple graphs in [KT19, HRW20] and with polylogarithmic error for weighted graphs in [Li21]. Extending the techniques in [KT19, HRW20] to weighted graphs presents significant challenges, and moreover, the algorithm can only polylogarithmically approximately preserve minimum cuts. A remaining challenge is to reduce the polylogarithmic-approximate clusterings to $1+o(1/\log n)$-approximate so that they can be applied recursively as in [Li21] over $O(\log n)$ many levels. This is an additional challenge that requires building on properties of tree-packings in the presence of a wide range of edge weights to, for example, find sources for local flow computations which identify minimum cuts that cross clusters.

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A graph and its ***two cuts***: the red dashed lines indicate a cut consisting of three edges, while the green lines represent a ***minimum cut of the graph***, consisting of two edges.

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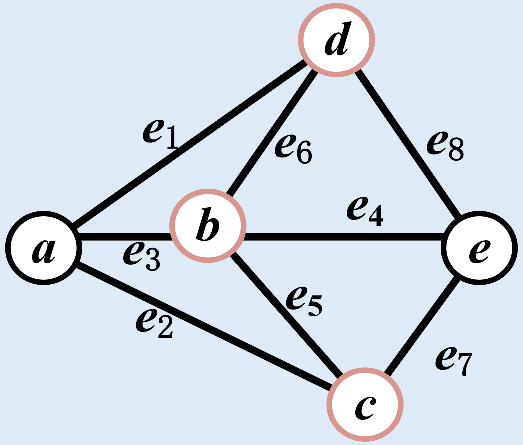
- **Definition 6.10:** An undirected connected graph ***G*=<*V*,*E*>**,
  - $\kappa(G)$=min{|*V'*| | *V'* is a vertex cut of *G or G-V'* becomes a trivial graph} is called the ***vertex connectivity*** of ***G***.
  - $\lambda(G)$=min{|*E'*| | *E'* is an edge cut of *G*} is called the ***edge connectivity*** of ***G***.

- **Example:**



$\kappa(G)=3$

$\lambda(G)=3$

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- **Notes:**

  (1) If G is a ***trivial graph***, then $\kappa(G)=0$, $\lambda(G)=0$.

  (2) If G is a ***complete graph K_n***, then $\kappa(G)=n-1$, $\lambda(G)=n-1$.

  (3) If G has a ***cut vertex***, then $\kappa(G)=1$; if G has a ***bridge*** (cut edge), then $\lambda(G)=1$.

  (4) By convention, the vertex connectivity and edge connectivity of a ***disconnected graph*** are both defined to be **0**.

- **Theorem 6.3:** For ***any undirected graph*** **G**, we have

$$\kappa(G) \le \lambda(G) \le \delta(G).$$

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↳ Theorem on the Connectivity of Directed Graphs

- **Let *D*=<*V*,*E*> be a directed graph, *u*,*v*∈*V*,**

(1) ***u*** **is** reachable from ***v***: There exists a path from ***u*** to ***v***. By convention, every vertex is reachable from itself.

(2) ***u*** **and** ***v*** are ***mutually reachable***: ***u*** is reachable from ***v***, and ***v*** is reachable from ***u***.

(3) ***D*** is ***weakly connected*** **(connected)**: The undirected graph obtained by ignoring the directions of all edges is connected.

(4) ***D*** is ***unilaterally connected***: ∀***u***,***v***∈***V***, either ***u*** is reachable from ***v*** or v is reachable from ***u***.

(5) ***D*** is ***strongly connected***: ∀***u***,***v***∈***V***, ***u*** and ***v*** are mutually reachable.

(6) ***D*** is ***strongly connected*** if and only if there ***exists a circuit*** that passes through all vertices.

(7) ***D*** is ***unilaterally connected*** if and only if there ***exists a path*** that passes through all vertices.

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- ***Shortest path*** **from** ***u*** **to** ***v***: The path from ***u*** to ***v*** with the minimum length (assuming ***u*** is reachable from ***v***).
- ***Distance d<u,v>***: The length of the shortest path from ***u*** to ***v***. If ***u*** is not reachable from ***v***, then by convention, **<u,v>=∞**.

- **Properties of a Distance Function *d<u,v>* :**
  - $d\langle u,v\rangle \geq 0$ and $d\langle u,v\rangle = 0 \Leftrightarrow u=v$
  - $d\langle u,v\rangle + d\langle v,w\rangle \geq d\langle u,w\rangle$
  - **Note:** Distance is ***not symmetric***

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Objective：

Key Concepts：

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- 6.3.1 Incidence Matrix of an Undirected Graph
- 6.3.2 Incidence Matrix of a Directed Acyclic Graph
- 6.3.3 Adjacency Matrix of a Directed Graph, Adjacency Matrix of an Undirected Graph

  The Number of Paths and Cycles in a Graph
- 6.3.4 Reachability Matrix of a Graph

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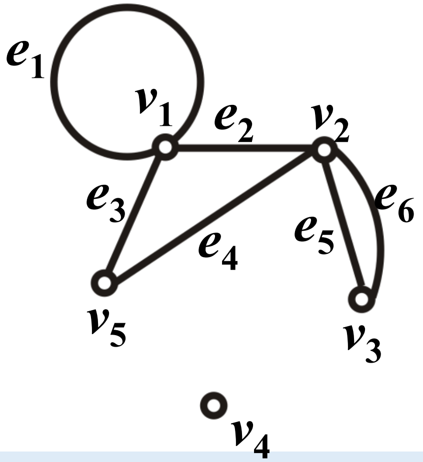
- Let G=<V,E>, V={v_1, v_2, …, v_n}, E={e_1, e_2, …, e_m}.
  - Let m_{ij} be the incidence of vertex v_i with edge e_j, the matrix $(m_{ij})_{n\times m}$ is called the ***incidence matrix*** of G, denoted as $M(G)$. The possible values of m_{ij}: 0,1,2

Example:

$$M(G)=\begin{bmatrix} 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$



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(1) $$\sum_{i=1}^{n} m_{ij} = 2, \quad j = 1, 2, \ldots, m$$

(2) $$\sum_{j=1}^{m} m_{ij} = d(v_i), \quad i = 1, 2, \ldots, n$$

(3) $$\sum_{i,j} m_{ij} = 2m$$

(4) e_j and e_k are ***parallel edge***
⇔ the j–th column and the k–th column are identical.

(5) V_i is an ***isolated vertex*** ⇔ i-th row is all zeros.

(6) E_j is ***loop*** ⇔ The first element in column j is 2, others are all 0.

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↳ The Incidence Matrix of a DAG

- Let the *directed acyclic graph* D=<V,E>, $V=\{v_1, v_2, \ldots, v_n\}$, $E=\{e_1, e_2, \ldots, e_m\}$.

  Let $m_{ij} = \begin{cases} 1, & v_i \text{ is the starting point of } e_j \\ 0, & v_i \text{ is not incident to } e_j \\ -1, & v_i \text{ is the endpoint of } e_j \end{cases}$

  The matrix $(m_{ij})_{n\times m}$ called the incidence matrix of D, called $M(D)$.

- **Properties of a DAG:**

  (1) **Each column** contains exactly one 1 and one −1.

  (2) The total number of **1's** is equal to the total number of **−1's**, which equals the number of edges.

  (3) The number of **1's** in the i-th row equals $d^+(v_i)$, and the number of −1's in the i-th row equals $d^-(v_i)$.

  (4) E_j *and* e_k are parallel edges ⇔ the **j-th** column and the **k-th** column are identical.

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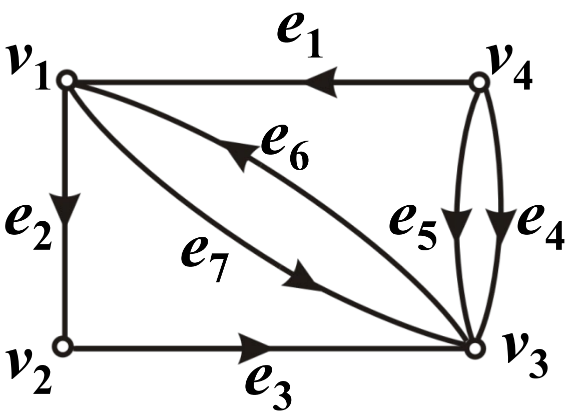
↳ The Incidence Matrix of a DAG(e.g.)



$$M(D)=\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

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- Let *D*=⟨*V*,*E*⟩, where $V=\{v_1, v_2, \ldots, v_n\}$ and $E=\{e_1, e_2, \ldots, e_m\}$, Let the number of edges from vertex v_i to vertex v_j be denoted as $a_{ij}^{(1)}$. The matrix $(a_{ij}^{(1)})_{m\times n}$ is called the ***adjacency matrix*** of *D*, denoted as *A*(*D*), or simply *A*.

- **Properties of *A*(*D*):**

(1) $$\sum_{j=1}^{n} a_{ij}^{(1)} = d^{+}(v_i), \quad i = 1,2,\ldots,n$$

The sum of the rows equals the outdegree of the graph.

(2) $$\sum_{i=1}^{n} a_{ij}^{(1)} = d^{-}(v_j), \quad j = 1,2,\ldots,n$$

The sum of the columns equals the indegree of the graph.

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- **Properties of A(D):**

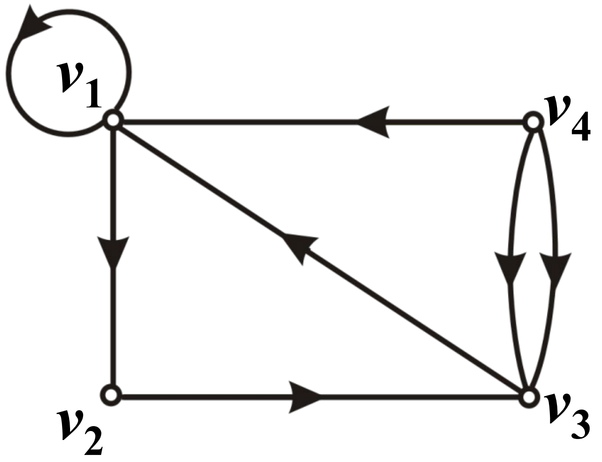
(3) $\sum_{i,j} a_{ij}^{(1)} = m$

The sum of all the elements equals the number of edges.

(4) $\sum_{i=1}^{n} a_{ii}^{(1)} = \textit{the number of loops at } D$

The sum of the diagonal elements equals the number of vertex loops.

- **Example:**



$$A=\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix}$$

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