■ **Example 4**: Prove that there does not exist a polyhedron with an odd number of faces such that each face has an odd number of edges.

✓ Such a polyhedron does not exist.

Prove:

①Suppose there are *n* faces, and each face has $a_1, a_2, \dots, a_n$ edges.

②If all these numbers are odd, then the sum $a_1+a_2+\dots+a_n$ is an odd number.

③However, each edge is shared by exactly two faces, which means each edge is counted twice in the sum above.

④Therefore, if there are *m* edges in total, then $2m= a_1+a_2+\dots+a_n$, this leads to an even number equaling an odd number — a ***contradiction***.

- **Example 5:** Suppose a simple undirected graph of order **9** has each vertex of degree **5** or **6**. Prove that it has **at least 5 vertices of degree 6** or **at least 6 vertices of degree 5**.

Proof 1: Consider all possible cases. Suppose there are a vertices of degree **5** and ***b*** vertices of degree **6**.

(1)a=0, b=9;

(2)a=2, b=7;

(3)a=4, b=5;

(4)a=6, b=3;

(5)a=8, b=1

(1)–(3): At least 5 vertices of degree 6

(4) and (5): At least 6 vertices of degree 5

Proof 2: Suppose $b<5$, then $a>9-5=4$. By a corollary of the Handshaking Lemma, $a\geq 6$.

■ **Definition 6.3:**

(1) In an ***undirected graph***, two or more edges connecting the same pair of vertices are called ***parallel edges***, and the number of such edges is called the **multiplicity**.

(2) In a ***directed graph***, two or more edges with the same starting and ending vertices are called ***directed parallel edges*** (or simply **parallel edges**), and their number is also referred to as the **multiplicity**.

(3) A graph that contains parallel edges is called a ***multigraph***.

(4) A graph that has neither parallel edges nor loops is called a ***simple graph***.

↳ Determining a Simple Graph (e.g.)

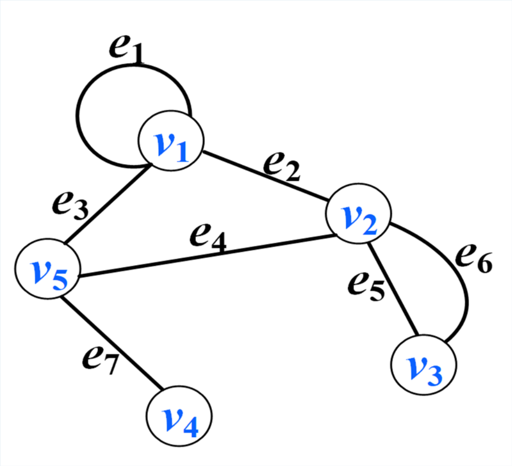


Figure 1

- Edges e_5 and e_6 are parallel edges.
- The multiplicity is **2**.
- It is not a simple graph.

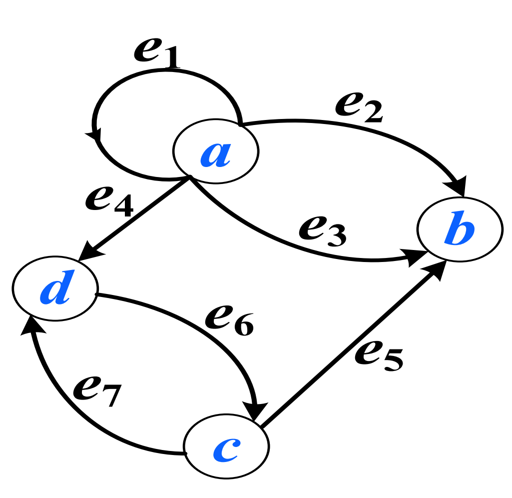


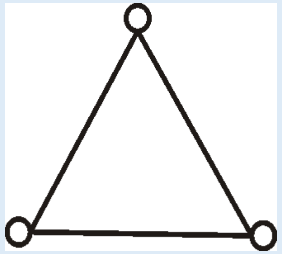
Figure 2

- Edges e_2 and e_3 are parallel edges, with a multiplicity of **2**.
- Edges e_6 and e_7 are not parallel edges.
- It is not a simple graph.

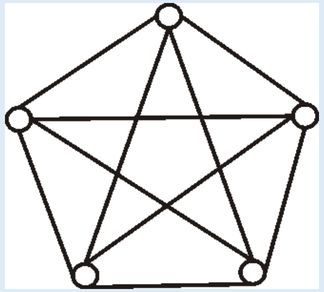
↳ Complete Graphs and Regular Graphs

- ***Undirected complete graph***: An undirected simple graph in which every pair of vertices is connected by an edge.
  - An undirected complete graph of order n is denoted by K_n, Number of vertices: n, Number of edge: $m=n(n-1)/2$, $\Delta=\delta=n-1$.
- ***Directed complete graph***: A directed simple graph in which every pair of vertices is connected by two edges in opposite directions.
  - Number of vertices: n, Number of edges: $m=n(n-1)$, $\Delta^+=\delta^+=\Delta^-=\delta^-=n-1$, $\Delta=\delta=2(n-1)$.
- ***k-regular graph***: An undirected simple graph in which every vertex has degree k.
  - Number of vertices: n, Number of edges: $m=kn/2$.

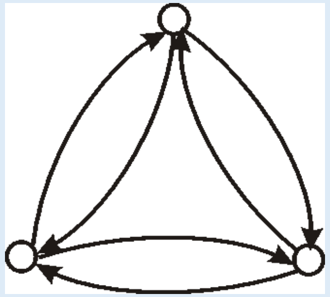
↳ Complete Graphs and Regular Graphs(e.g.)



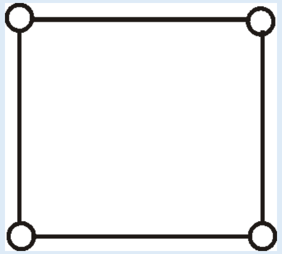
K_3



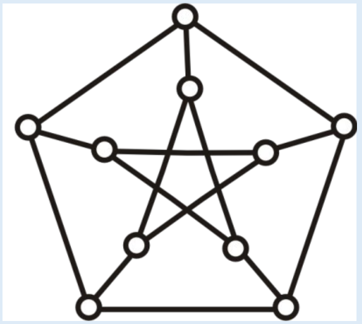
K_5



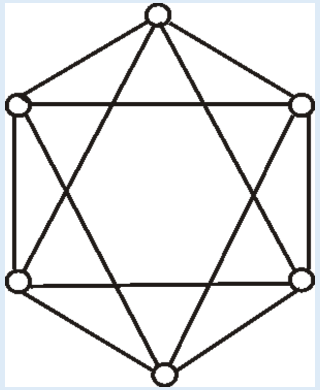
3-vertex directed complete graph



2-regular undirected graph

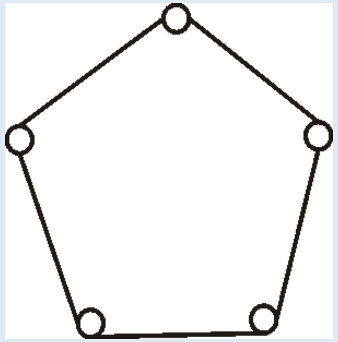


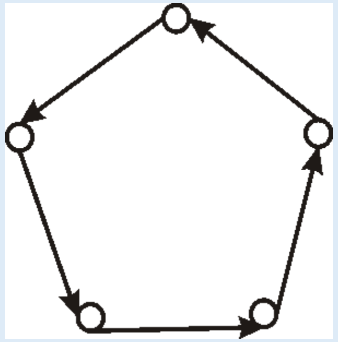
3-regular graph Petersen graph

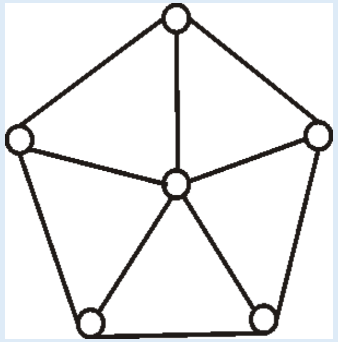


4-regular undirected graph

- *Undirected cycle graph* C_n=<V,E>, where $V=\{v_1,v_2,\ldots,v_n\}$, $E=\{(v_1,v_2),(v_2,v_3),\ldots,(v_{n-1},v_n),(v_n,v_1)\}$, $n \geq 3$, E is a set of **unordered pairs.**
- *Directed cycle graph* D_n=<V,E>, where $V=\{v_1,v_2,\ldots,v_n\}$, $E=\{<v_1,v_2>, <v_2,v_3>,\ldots,<v_{n-1},v_n>,<v_n,v_1>\}$, $n \geq 3$, E is a set of **ordered pairs.**
- *Wheel graph* W_n: add a single vertex to the undirected cycle graph C_{n-1} and connect it to every vertex of the cycle with exactly one edge, $n \geq 4$.
- **Example:**





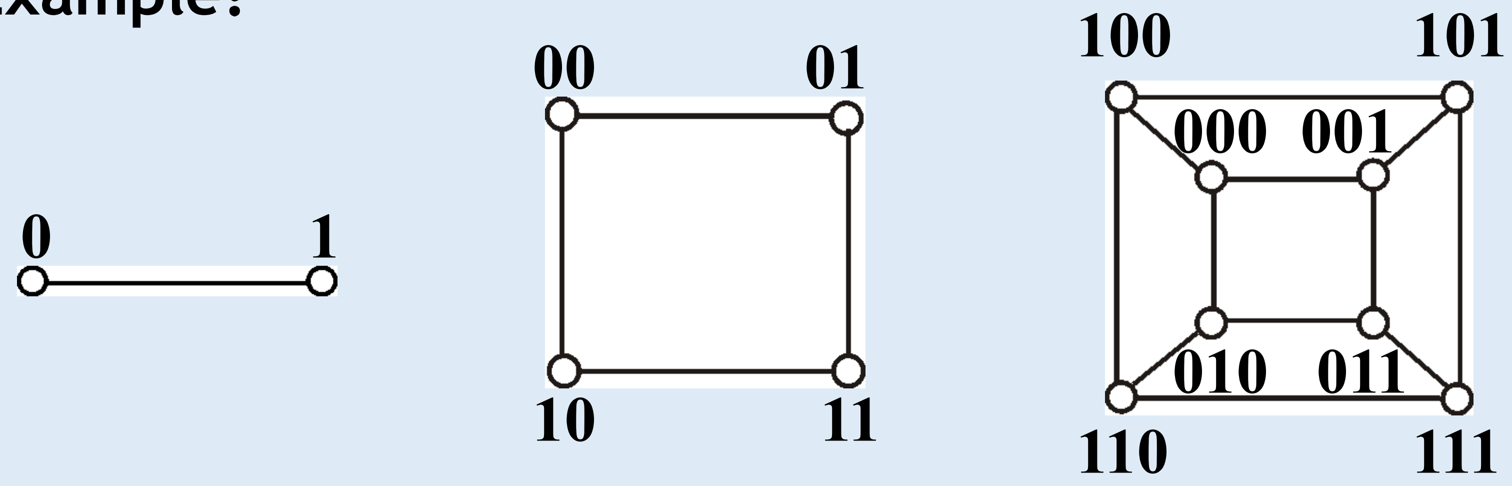


- The *n-dimensional hypercube graph* Q_n=<V,E> is a simple undirected graph of order 2^n, where

  $V=\{v \mid v=a_1a_2\ldots a_n,\ a_i=0,1,\ i=1,2,\ldots,n\}$

  $E=\{(u,v) \mid u,v \in V \wedge u$ *and* v differ in exactly one bit position}.

- **Example:**



- **Definition 6.4: Let $G=<V,E>$, $G'=<V',E'>$** be two graphs (both undirected or both directed).

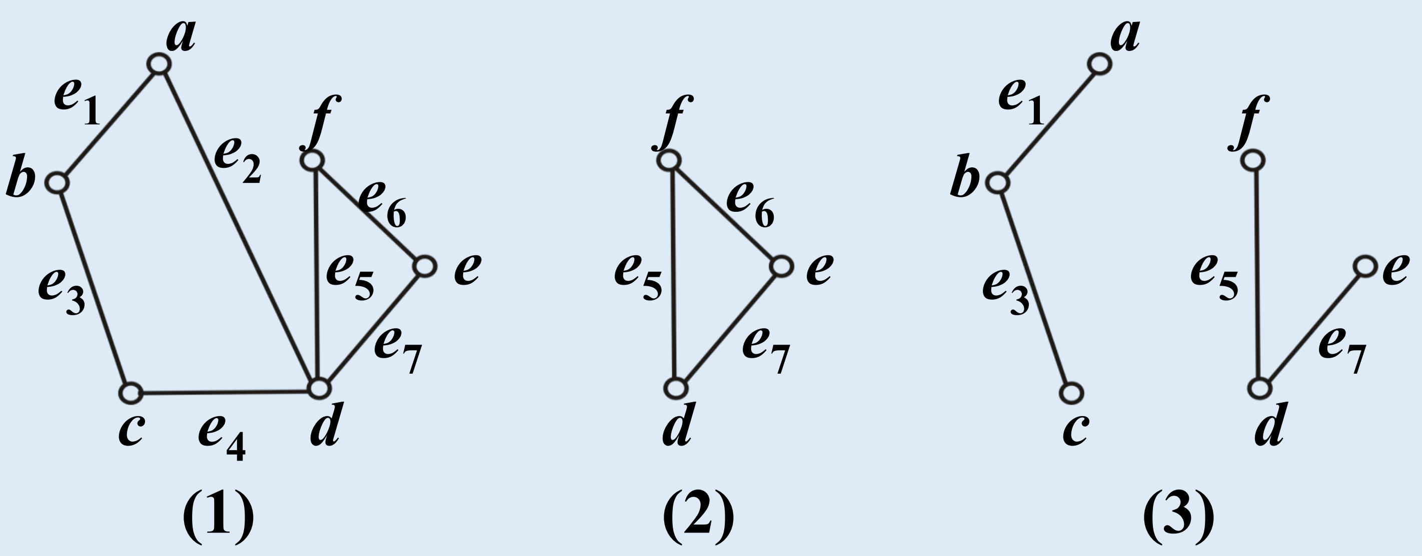
(1) ***If*** $V'\subseteq V$ **and** $E'\subseteq E$, **then** G' is G's ***subgraph*** , G is called G's ***supergraph***, **denoted as** $G'\subseteq G$.

(2) ***If*** $G'\subseteq G$ ***and*** $V'=V$, **then** G' is called a ***spanning subgraph*** of G.

(3) ***If*** $V'\subset V$ ***or*** $E'\subset E$, **then** G' is called a ***proper subgraph*** of G.

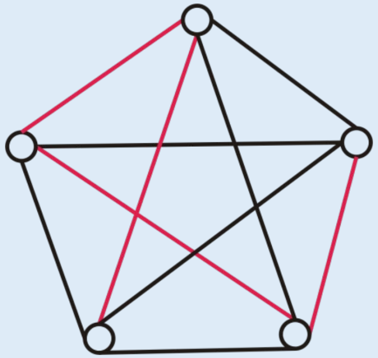
(4) ***Let*** $V'\subseteq V$ ***and*** $V'\neq\varnothing$, the subgraph of G with vertex set V' and edge set consisting of all edges in G whose endpoints are both in V' is called the ***vertex-induced subgraph*** on V' ,denoted by $G[V']$.

(5) ***Let*** $E'\subseteq E$ **and** $E'\neq\varnothing$, the subgraph of G with edge set E' and vertex set consisting of all endpoints of the edges in E' is called the ***edge-induced subgraph*** on E' ,denoted by $G[E']$.



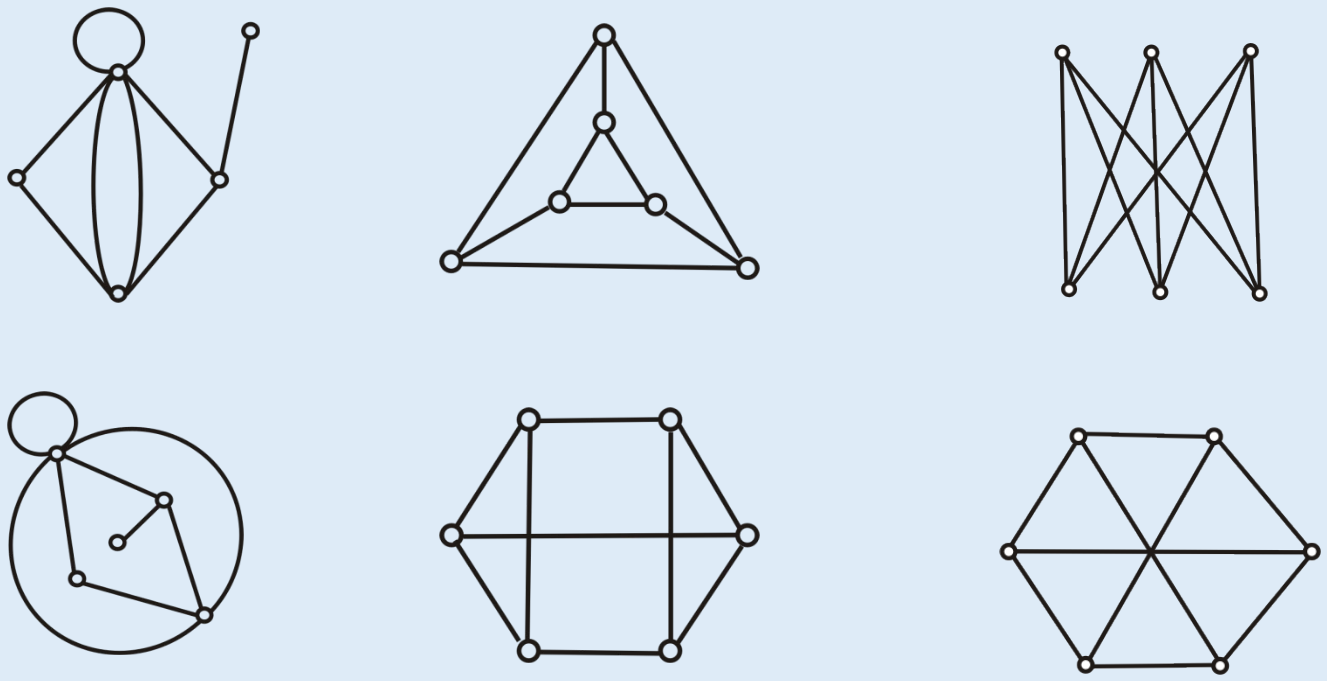
- (1),(2),(3) is (1) subgraphs , (2),(3) are proper subgraphs, (1) is the supergraph.
- (1),(3) are spanning subgraphs of (1).
- (2) is the vertex-induced subgraph on $\{d,e,f\}$, and also the edge-induced subgraph on $\{e_5, e_6, e_7\}$.
- (3) is the edge-induced subgraph on $\{e_1, e_3, e_5, e_7\}$.

- **Definition 6.5:** Let ***G*=<*V*,*E*>** be a simple undirected graph of order ***n***, let $\overline{\boldsymbol{E}} = \boldsymbol{V}\&\boldsymbol{V} - \boldsymbol{E}$, then $\overline{\boldsymbol{G}} =< \boldsymbol{V}, \boldsymbol{E} >$ is called the ***complement*** of graph ***G***.

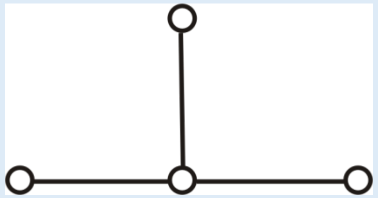


- **Definition 6.6:** Let G_1=<V_1,E_1>, G_2=<V_2,E_2> be two undirected graphs (or two directed graphs), if there exists a bijective function f: $V_1 \to V_2$, such that for any $v_i, v_j \in V_1$:
  - $(v_i,v_j) \in E_1$ if and only if $<f(v_i), f(v_j)> \in E_2$
  - and the multiplicity of (v_i,v_j) **is equal to** $(f(v_i),f(v_j))$
  - then G_1 **and** G_2 are said to be ***isomorphic***, denoted by $G_1 \cong G_2$.

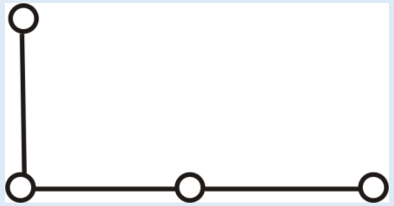
↳ Isomorphic graphs（e.g.）



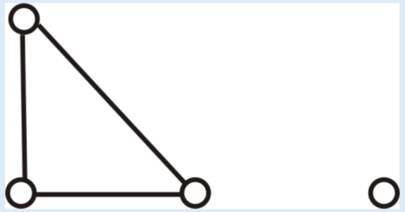
- **Example :** Draw all non-isomorphic simple undirected graphs with **4** vertices and **3** edges.
- **Solution:**
  - The total degree is **6** (since the sum of degrees equals twice the number of edges). This must be distributed among **4** vertices.
  - The maximum degree is **3**, and the number of vertices with odd degree must be even.
  - There are three possible degree sequences:**1, 1, 1, 3**； **1, 1, 2, 2**； **0, 2, 2, 2**



1,1,1,3

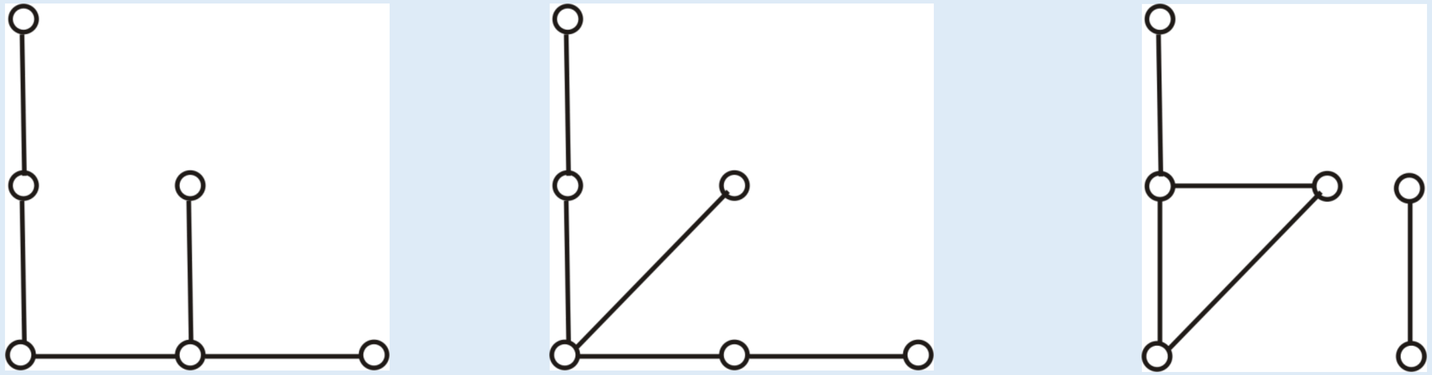


1,1,2,2



0,2,2,2

- **Example**: Draw three non-isomorphic simple undirected graphs with the degree sequence **1,1,1,2,2,3.**



Objective :

Key Concepts :