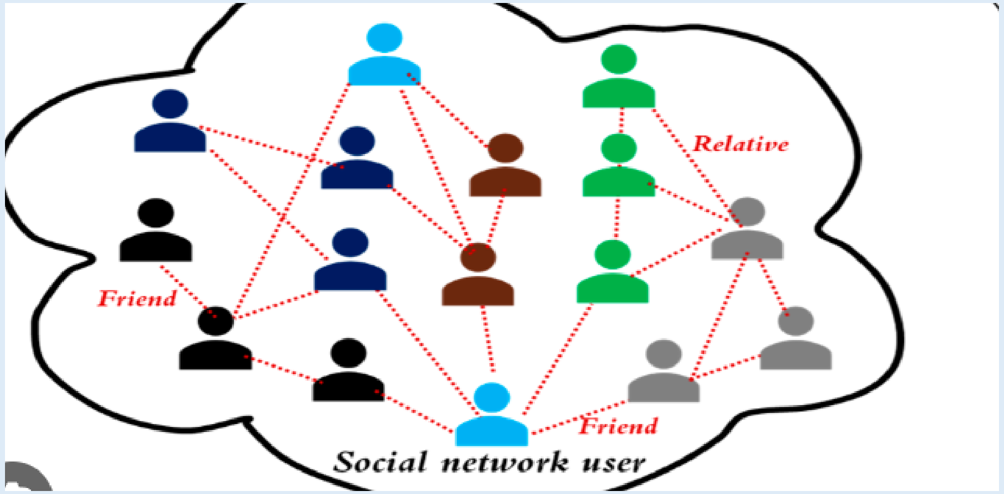
Discrete Mathematics 2025 Spring

魏可佶 kejiwei@tongji.edu.cn

- 6.1 Basic Concepts of Graphs
- 6.2 Graph Connectivity
- 6.3 Matrix Representations of Graphs
- 6.4 Special Types of Graphs

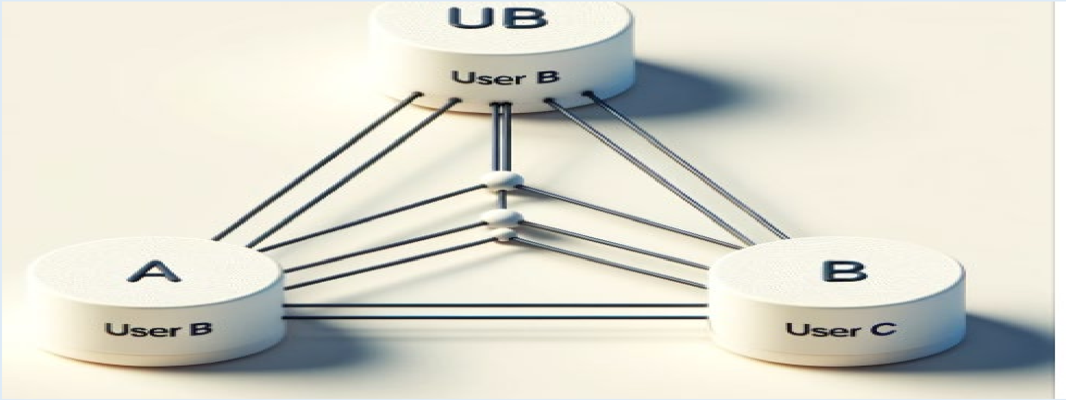
Social Networks

We can use a simple graph to represent whether two people know each other.



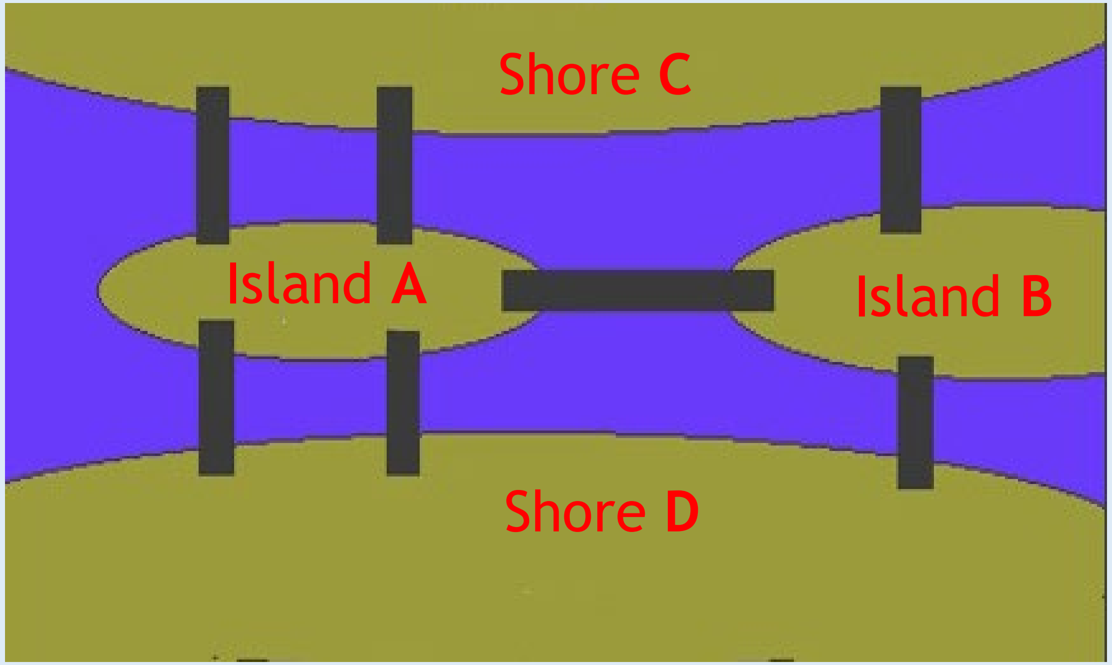
Communication networks

We can use vertices to represent devices and edges to represent the type of communications link of interest.



Transportation Networks

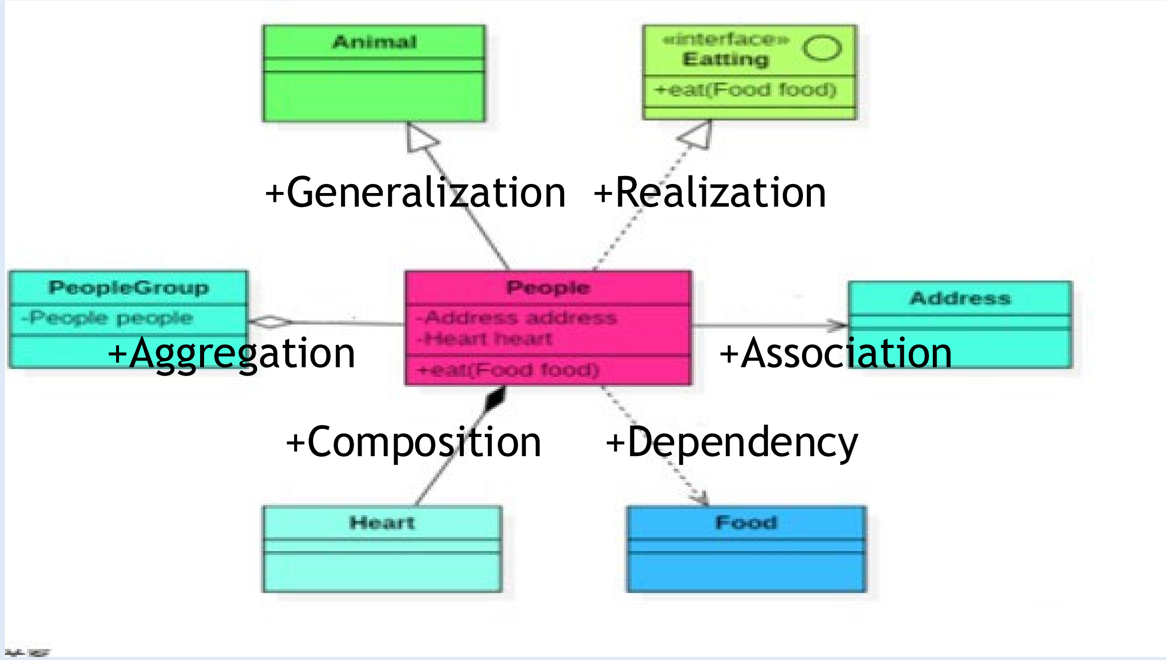
We can use a simple graph to represent roads, air and rail networks, etc...



Seven Bridges of Königsberg

Software design applications

We can use a simple graph to represent how different blocks interact each other.



- **Shortest Path Problem:**
  Given a weighted graph, find the shortest path between two nodes.
  This problem can be solved using algorithms such as **Dijkstra's algorithm**, **Bellman-Ford algorithm**, and **Floyd-Warshall algorithm.**
- **Minimum Spanning Tree Problem:**
  Given a weighted, undirected, and connected graph, find a spanning tree with the minimum total edge weight.
  This problem can be solved using algorithms such as **Kruskal's algorithm** and **Prim's algorithm.**
- **Maximum Flow Problem:**
  In a directed graph, find the maximum possible flow from a source node to a sink node.
  This problem can be solved using algorithms such as **Ford-Fulkerson algorithm** and **Dinic's algorithm.**

- **Topological Sorting Problem:**

  Given a directed acyclic graph (DAG), arrange its nodes in a linear sequence such that all directed edges go from earlier to later nodes in the sequence.

  This problem can be solved using **depth-first search (DFS)** and **breadth-first search (BFS)**.

- **Minimum Cut Problem:**

  In a weighted undirected graph, find a set of edges whose removal disconnects the graph and whose total weight is minimized.

  This problem can be solved using algorithms such as the **Stoer-Wagner algorithm** and **Karger's algorithm**.

↳ Unordered Product(&) and Set Multiplicity

- **Unordered Pair:** A set consisting of two elements, denoted as $\{a,b\}$.
- **Unordered Product:** $A\&B=\{(x,y) \mid x\in A\wedge y\in B\}$
- **Example:** $A=\{a,b,c\}$, $B=\{1,2\}$

  $A\&B=B\&A=\{(a,1), (b,1), (c,1), (a,2), (b,2), (c,2)\}$

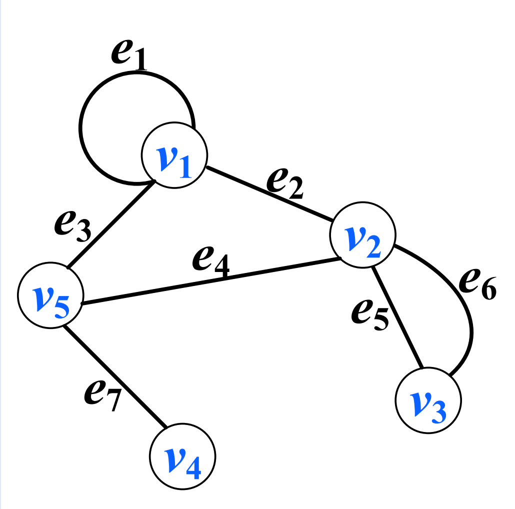
  $A\&A=\{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c)\}$

  $B\&B=\{(1,1), (1,2), (2,2)\}$
- **Multiset:** A set in which elements are allowed to appear more than once.
- **Multiplicity:** The number of times an element appears in a multiset.
- **Example:** $S=\{a,b,b,c,c,c\}$, Then the multiplicities of a,b,c is 1,2,3

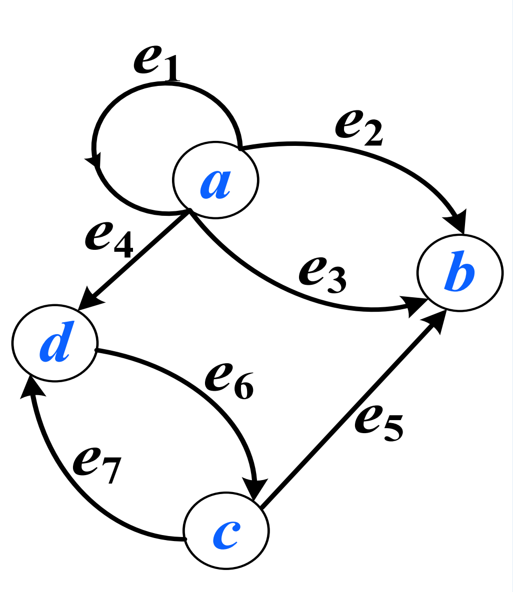
- **Definition6.1:** An *undirected graph* $G=\langle V,E\rangle$, where $V\neq\varnothing$ is called the vertex set, and its elements are called **vertices** or **nodes.** E is a **multisubset** of the unordered product $V\&V$, and is called the edge set, whose elements are **undirected edges**, or simply **edges**.
  - Sometimes, we use $V(G)$ and $E(G)$ to represent the vertex set and edge set of graph G, respectively.
- **Example:** $G=<V,E>$as shown in the figure:

Where $V=\{v_1, v_2, \ldots, v_5\}$

$E=\{(v_1,v_1), (v_1,v_2), (v_1,v_5), (v_2,v_3), (v_2,v_3), (v_2,v_5), (v_4,v_5)\}$



- **Definition6.2:** A ***directed graph*** is denoted by $D=\langle V,E\rangle$, where $V\neq\varnothing$ is called the **vertex set**, and its elements are called **vertices** or **nodes**. E is a **multisubset** of the Cartesian product $V\times V$, and is called the **edge set**, whose elements are **directed edges**, or simply **edges.**
  - Sometimes, we use $V(D)$ and $E(D)$ to represent the vertex set and edge set of the graph D, respectively.



- **Finite Graph:** A graph in which both V, E are finite sets.
- **n-Vertex Graph (Graph of Order n):** A graph with exactly n vertices.
- **Null Graph:** A graph with $E=\varnothing$.
- **Trivial Graph:** A graph with only one vertex and no edges.
- **Empty Graph:** A graph with $V=\varnothing$

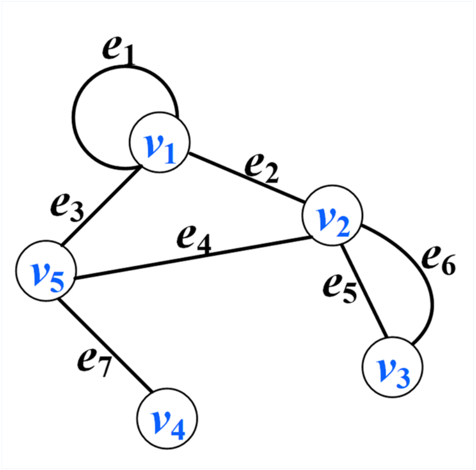
- ***Let*** $G=\langle V,E\rangle$ be an undirected graph ***and let*** $e_k=(v_i, v_j)\in E$, **then** v_i, v_j *are the end point of edge* e_k, e_k *is said to be incident to* v_i (v_j).
  - *If* $v_i \neq v_j$, the **incidence number** of e_k *with respect* to v_i (v_j) is 1.
  - **If** $v_i = v_j$, the incidence number of e_k *with respect to* v_i is 2.
  - **If** v_i is not an endpoint of edge e, then the incidence number of e with respect to v_i is 0.
- **Let** $v_i,v_j\in V$, $e_k,e_l\in E$, *if*$(v_i,v_j)\in E$, **then** v_i,v_j ***adjacent***.
  - *If* e_k,e_l share a common endpoint, then they are said to be ***adjacent edges.***

- Let directed gragh $D=<V,E>$, $e_k=\langle v_i,v_j\rangle\in E$, then v_i, v_j are called ***endpoint*** *of* e_k, v_i is e_k ***start vertex***, v_j is e_k ***end vertex***.
  - The edge e_k is said to be incident from v_i (v_j).
  - If the head of edge e_k is the tail of edge e_l, then e_k and e_l are said to be ***adjacent edges***.

- In both undirected and directed graphs, an edge that connects a vertex to itself is called a *loop*.A vertex with no incident edges is referred to as an ***isolated vertex***.

↳ The Degree of a Vertex in an Undirected G

- Let $G=<V,E>$ be an undirected graph, $v \in V$,
  - The ***degree $d(v)$*** of a vertex v: the number of edges incident to v.
  - A ***pendant vertex***: a vertex with degree 1.
  - A ***pendant edge***: an edge that is incident to a pendant vertex.
  - The ***maximum degree*** of G: $\Delta(G)=\max\{d(v) \mid v \in V\}$.
  - The ***minimum degree*** of G: $\delta(G)=\min\{d(v) \mid v \in V\}$.
- **For example :**



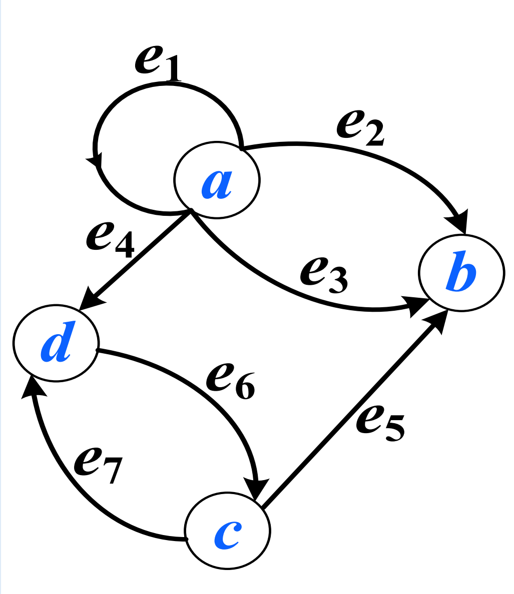
$d(v_5)=3$, $d(v_2)=4$, $d(v_1)=4$,
$\Delta(G)=4$, $\delta(G)=1$,
v_4 is a pendant vertex, e_7 is a pendant edge, e_1 is a loop.

↳ The Degree of a Vertex in an directed Graph

- **Let $D=<V,E>$ to be a directed gragh, $v \in V$,**
  - The *out-degree $d^+(v)$*: the number of edges where **v** is the starting point.
  - The *in-degree $d^-(v)$*: the number of edges where v *is the ending point*.
  - The *degree $d(v)$* of a vertex **v** : the total number of edges incident to **v** (as either the starting or ending point). **$d(v)= d^+(v)+ d^-(v)$.**
- Maximum and minimum out-degree in **D**: $\Delta^+(D)$, $\delta^+(D)$
- Maximum and minimum in-degree in **D**: $\Delta^-(D)$, $\delta^-(D)$
- Maximum and minimum total degree in **D**: $\Delta(D)$, $\delta(D)$
- **Example:** $d^+(a)=4$, $d^-(a)=1$, $d(a)=5$,

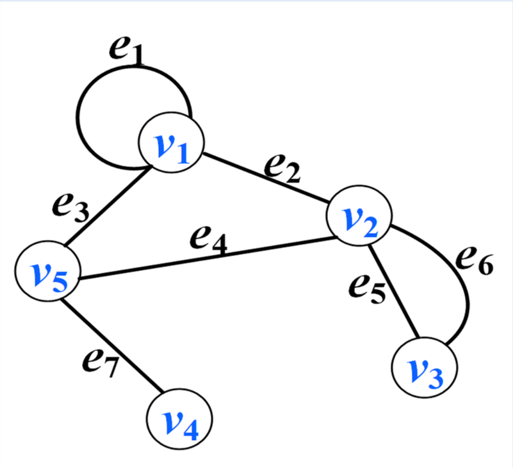
  $d^+(b)=0$, $d^-(b)=3$, $d(b)=3$,

  $\Delta^+=4$, $\delta^+=0$, $\Delta^-=3$, $\delta^-=1$, $\Delta=5$, $\delta=3$



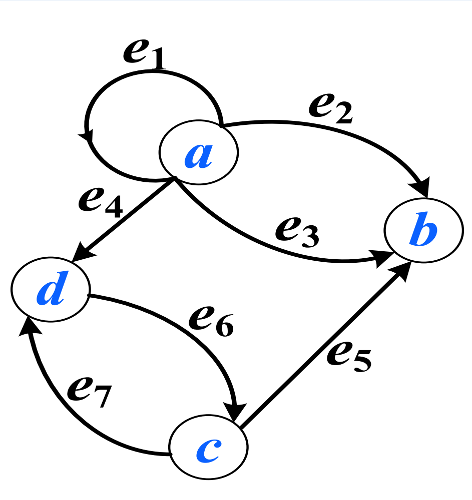
- **Theorem 6.1:** In any graph (**undirected or directed**), the ***sum of the degrees*** of all vertices is equal to twice the number of edges($\sum_{i=1}^{n} d(v_i)=2m$).
  - **Proof:** Each edge in the graph (including loops) has two endpoints. Therefore, when summing the degrees of all vertices, each edge contributes 2 to the total degree.
  - With m edges, the total degree is $\sum d(v)=2m$.
- **Corollary:** Any graph (undirected or directed) has an ***even number of vertices*** with ***odd degree***.
- **Theorem 6.2:** In a directed graph, the sum of the in-degrees of all vertices equals the sum of the out-degrees, and both are equal to the number of edges($\sum_{i=1}^{n} d^{+}(v_i)=\sum_{i=1}^{n} d^{-}(v_i)=m$).
  - **Proof:** Each edge contributes exactly one in-degree and one out-degree.

- Let the vertex set of an **undirected graph *G*** be $V=\{v_1, v_2, \ldots, v_n\}$.
  - The ***degree sequence*** of G: $d(v_1), d(v_2), \ldots, d(v_n)$
- **Example:**
  The degree sequence of Figure 1 is: **4, 4, 2, 1, 3.**



(**Figure 1**)

- Let the vertex set of a **directed graph *D*** be $V=\{v_1, v_2, \ldots, v_n\}$
  - The ***degree sequence of D*** *is*: $d(v_1), d(v_2), \ldots, d(v_n)$
  - The ***out-degree sequence*** of *D is*: $d^+(v_1), d^+(v_2), \ldots, d^+(v_n)$
  - The ***in-degree sequence*** of *D is*: $d^-(v_1), d^-(v_2), \ldots, d^-(v_n)$
- **Example:** In the Figure 2 :
  - Degree sequence :**5,3,3,3**
  - Out-degree sequence:**4,0,2,1**
  - In-degree sequence:**1,3,1,2**



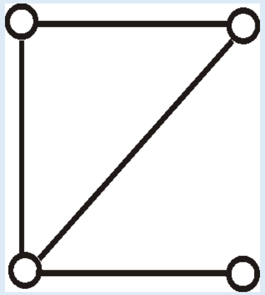
(**Figure 2**)

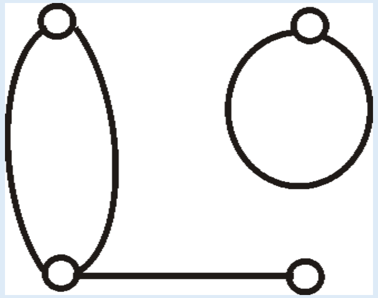
- **Example 1：** Can the following two sets of numbers be the degree sequences of an undirected graph?

  (1) 3,3,3,4; (2) 1,2,2,3

Solve: (1) No. odd numbers odd degree.

(2) Yes.





- **Example 2**: Given that graph ***G*** has **10** edges and **4** vertices of degree **3**, and the degrees of all remaining vertices are less than or equal to **2**, what is the minimum number of vertices in ***G***？
- **Solution**: Suppose graph ***G*** has ***n*** vertices. By the Handshaking Lemma, if n people each shake hands ***x*** times, the total number of handshakes is $S = nx/2$。

  $4\times3+2\times(n-4)\geq2\times10$ ， Solve $n\geq8$
- **Example 3**: Given a directed graph of order **5**, the degree sequence and out-degree sequence are respectively **3,3,2,3,3**， and **1,2,1,2,1**. Find its in-degree sequence.

  Solve **2,1,1,1,2**