Discrete Mathematics 2025 Spring

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- 5.1 Function Definition and Properties
- 5.2 Composition of Functions and Inverse Functions
- 5.3 Relational Algebra

5.1.1 Definition of a Function

- Equality of functions
- Function's domain, range, and direction
- Surjective, Injective, and Bijective Functions
- Function Sets and Counting
- Constant, Identity, and Monotonic Functions
- Natural mapping
- Equivalence relation’s impact on natural mapping
- The order of a complexity function

5.1.2 Image and Preimage of a Function

5.1.3 Properties of a Function

- **Definition 5.1:** *function*

  Let f be a binary relation. If for every $\forall x \in \mathbf{dom}f$ there exists a unique $y \in \mathbf{ran}f$ such that $x\ f\ y$ holds, then f is called a ***function***. For a function f, if $x\ f\ y$, we denote this as $y=f(x)$, and y s called the value of ***f at x***.

- **For example:**

$$f_1=\{<x_1,y_1>,<x_2,y_2>,<x_3,y_2>\}$$

$$f_2=\{<x_1,y_1>,<x_1,y_2>\}$$

f_1 is a function, but f_2 is not a function.

- **Definition 5.2:** Equality of functions

  Let f, g be functions. Then, f, g are equal if and only if their set representations are equal: $f=g \Leftrightarrow f\subseteq g \wedge g\subseteq f$

- If two functions f *and* g are equal, the following two conditions must be satisfied:

  (1) dom$f$ = domg

  (2) $\forall x \in$ dom$f$ = domg

  we have $f(x) = g(x)$

- **Example:** $f(x)=(x^2-1)/(x+1)$, $g(x)=x-1$

  These functions are not equal because **dom$f\subset$domg.**

- **Definition 5.3:** function f:$A \to B$

  Let A and B be sets. If

  (1) f is a function,

  (2) dom$f$ = A *and*

  (3) ran$f \subseteq B$,

  then f is called a **function from** ***A to B***, denoted by $f : A \to B$.

- **Examples:**

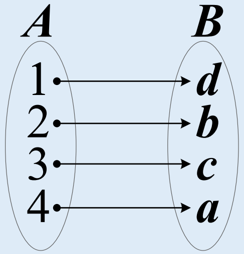
  $f : \mathbf{N} \to \mathbf{N}$, $f(x)=2x$ is a function from **N** ***to*** **N**

  $g : \mathbf{N} \to \mathbf{N}$, $g(x)=2$ is also a function from **N to N**

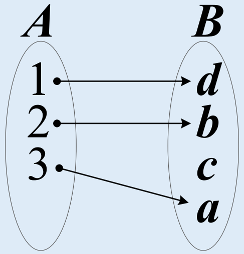
5.1.1 Definition of a Function

↳ Surjective, Injective, and Bijective Functions

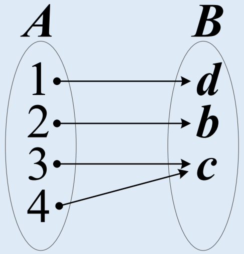
- **Definition 5.4:** Let $f : A \to B$
  - A function $f{:}A\to B$ is ***subjective (onto)*** if and only if for every $b\in B$, there exists an $a\in A$ such that $f(a)=b$.
  - function $f{:}A\to B$ is ***injective*** if and only if for all $a,b\in A$, we have $f(a)= f(b) \Rightarrow a=b$.
  - If $f{:}A\to B$ is both **surjective** and **injective**, then it is called a ***bijective*** function (or **bijection**).
- **Examples:**



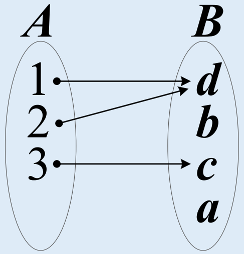
Bijective (Injective and Surjective)



Injective but not Surjective



Surjective but not Injective



Neither Surjective nor Injective

- **Definition 5.5:** The set of all functions from ***A to B*** is denoted by B^A, **read as "*B* to the power of *A*"**

  In symbolic form:

$$B^A = \{ f \mid f : A \to B \}$$

- **Counting B^A :**
  - $|A|=m$, $|B|=n$, **and** m, $n>0$, $|B^A|=n^m$.
  - $A=\varnothing$, **then** $B^A=B^{\varnothing}=\{\varnothing\}$.

    The function set contains only one element: the empty function. $|B^{\varnothing}|=1$
  - $A\neq\varnothing$ **and** $B=\varnothing$, **then** $B^A=\varnothing^A=\varnothing$.

    here is no function from a non-empty set to the empty set. $|\varnothing^A|=0$

- **Example: Let A = {1, 2, 3}, B = {a, b}, solve B^A.**

Solve: Find all possible functions from A to B.

$B^A = \{ f_0, f_1, \dots, f_7 \}$, then

$f_0=\{<1,a>,<2,a>,<3,a>\}$

$f_1=\{<1,a>,<2,a>,<3,b>\}$

$f_2=\{<1,a>,<2,b>,<3,a>\}$

$f_3=\{<1,a>,<2,b>,<3,b>\}$

$f_4=\{<1,b>,<2,a>,<3,a>\}$

$f_5=\{<1,b>,<2,a>,<3,b>\}$

$f_6=\{<1,b>,<2,b>,<3,a>\}$

$f_7=\{<1,b>,<2,b>,<3,b>\}$

Definition 5.6:

(1) Let f: $A \to B$, If there exists a constant $c \in B$ such that for all $x \in A$, $f(x)=c$, **then** $f : A \to B$ is called a ***constant function***.

(2) The **identity relation** I_A on A is called the ***identity function*** on A, where for all $x \in A$, $I_A(x)=x$.

(3) Let $<A, \preccurlyeq>$, $<B, \preccurlyeq>$ be partially ordered sets, $f : A \to B$ called ***monotonically increasing*** (or simply **monotonic**) if for any $x_1, x_2 \in A$, $x_1 \prec x_2$, $\Rightarrow f(x_1) \preccurlyeq f(x_2)$;

- ***strictly monotonically increasing*** if for any x_1, $x_2 \in A$, $x_1 \prec x_2$, , $\Rightarrow f(x_1) \prec f(x_2)$.
- Similarly, **monotonically decreasing** and **strictly monotonically decreasing** functions can be defined in the same manner.

(4) Let *A* be a set. For any subset $A' \subseteq A$, the ***characteristic function*** $\chi_{A'}: A \to \{0,1\}$ is defined as follows:

$$\chi_{A'}(a) = \begin{cases} 1, & a \in A' \\ 0, & a \in A - A' \end{cases}$$

- **Example**: **let** $A=\{a,b,c\}$, Each subset *A'* of *A* corresponds to a **characteristic function**, and different subsets correspond to different characteristic functions. **Such as** :

$$\chi_{\varnothing} = \{\langle a,0\rangle,\langle b,0\rangle,\langle c,0\rangle\}$$
$$\chi_{\{a,b\}} = \{\langle a,1\rangle,\langle b,1\rangle,\langle c,0\rangle\}$$

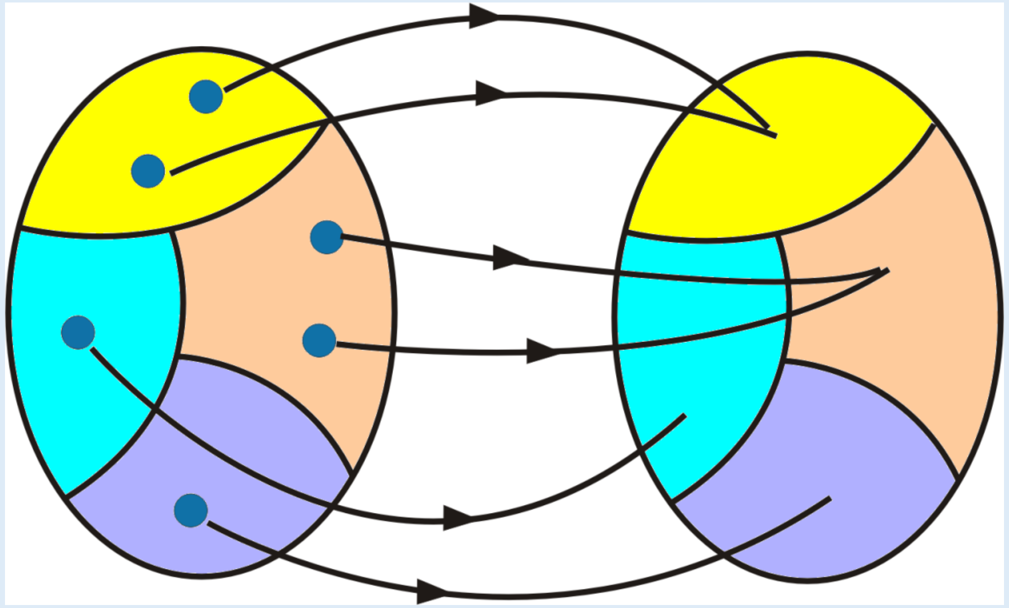
- The characteristic function of a set is a **detector** that determines whether an element belongs to the set; it is also known as an **indicator function** that represents whether an event occurs.

(5) Let R be an equivalence relation on A . Define

$$g : A \to A/R$$

$$g(a) = [a],\ \forall a \in A$$

Then g is called the ***natural mapping*** **(function)** from A to the quotient set A/R.



- Given a set A and an equivalence relation ***R on A***, a ***natural mapping*** g : $A \rightarrow A/R$ can be determined.
- Natural mappings vary with the equivalence relation:
  - The **identity relation** yields a **bijection.**
  - Others equivalence relations are generally **surjections** only.
- **Example**: Let $A=\{1, 2, 3\}$,
  - **Equivalence relation** : $R_1=\{<1,2>,<2,1>\} \cup I_A$

    Natural mapping : $g_1(1) = g_1(2) = \{1,2\}$, $g_1(3) = \{3\}$
  - **Equivalence relation**: I_A

    Natural mapping: $g_2(1)=\{1\}$, $g_2(2)=\{2\}$, $g_2(3)=\{3\}$

↳ The order of a complexity function

- $W : Z^+ \to Z^+$ be the ***time complexity function*** of an algorithm. It is a discrete, monotonically increasing function defined on positive integers.
- The meaning of **$W(n)$:** For an input of size **n**, the number of basic operations executed by the algorithm in the **worst case** is **$W(n)$.**
- **Asymptotic notation** for the order of a complexity function $f(n)$ ：

  $f(n)=O(g(n)) \Leftrightarrow f(n)$ is **no more than** that of $g(n)$

  $f(n)=\Theta(g(n)) \Leftrightarrow f(n)=O(g(n))$ **and** $g(n)=O(f(n))$
- **Examples:** $f(n)=n^2+n=\Theta(n^2)$, $g(n)=n\log n=O(n^2)$ (Here, **log*n*** is shorthand for **$\log_2 n$)**

  Algorithms：Binary search: $W(n)=O(\log n)$

  Merge sort: $W(n)=O(n\log n)$